Adjustments to Power Series

Given a power series $\sum_{n=0}^{\infty} a_n x^n$ with radius of convergence r,

there are various adjustments or replacements we can perform to obtain new series (and functions represented). Most do not change the radius if convergence, but two do:

(1) Replace the variable "x" with "kx", where k is any non-zero constant. The radius of convergence becomes $\frac{r}{|k|}$

reason: Look at the RATIO TEST on the original series

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = |x|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

(the series converges if this limit is < 1, so in order that the radius of convergence be r, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{r}$)

Now, replace "x" with "kx", and we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}(kx)^{n+1}}{a_n (kx)^n}\right| = |kx|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |kx|\frac{1}{r} < 1$$

so $|k||x| < r \quad \therefore |x| < \frac{r}{|k|}$ (this is the NEW radius of convergence)

Example 1: We know the radius of convergence of the series below is 1, and it converges for all x such that $-1 < x < 1$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots + x^n + \ldots$$

Replace "x" by "4x", and we obtain the function (and its series):

$$\frac{1}{1-4x} = 1 + 4x + 16x^2 + 64x^3 + 256x^4 + \ldots + (4x)^n + \ldots$$

(this new series has a radius of convergence 1/4; that is, the series represents the function for all x such that $-\frac{1}{4} < x < \frac{1}{4}$)

Alternate view: You could also think of the function $\frac{1}{1-4x}$ *as the sum of a geometric series with first term 1, and common ratio 4x*

Other adjustments (and their effect on radius of convergence):

(2) Replace the variable "x" with any positive power of x ... " x^N "
The radius of convergence becomes $\sqrt[N]{r}$ (where r is the original radius of convergence of $\sum a_n x^n$)

Reason: Use the same argument pattern as in (1)

Example: Consider the function $f(x) = \dfrac{1}{1-(3x)^5}$

What series represents this function, and what is its radius of convergence?

Solution: **Start with the function** $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots x^n + \ldots$

This series has radius of convergence 1.

Now, replace "x" by "$(3^5)x$" ... the new series is

$$\frac{1}{1-(3^5)x} = 1 + (3^5)x + ((3^5)x)^2 + ((3^5)x)^3 + ((3^5)x)^4 + \ldots ((3^5)x)^n + ..$$

(radius of convergence $1/(3^5)$ ... converges over $(-1/(3^5), 1/(3^5))$)

Next, replace "x" in this series with " x^5 " You obtain:

$$\frac{1}{1-(3x)^5} = 1 + (3x)^5 + (3x)^{10} + (3x)^{15} + (3x)^{20} + \ldots (3x)^{5n} + \ldots$$

(radius of convergence is now $\sqrt[5]{(1/(3^5))}$, or simply 1/3)

(3) Multiply entire series by a constant $k \neq 0$
(No change in the radius of convergence)

$$k\,f(x) = k\sum_{n=0}^{\infty} a_n x^n = ka_0 + ka_1x + ka_2x^2 + ka_3x^3 + ka_4x^4 + \ldots + ka_nx^n + \ldots$$

Example: $\dfrac{5}{1-x} = 5\displaystyle\sum_{n=0}^{\infty} x^n = 5 + 5x + 5x^2 + 5x^3 + 5x^4 + \ldots + 5x^n + \ldots$

(still converges over (-1 , 1)

(4) Multiply entire series by a non-zero function of x... h(x).
(No change in the radius of convergence)

$$h(x)\sum_{n=0}^{\infty} a_n x^n = h(x)(a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots)$$

Example: $\dfrac{\sin(x)}{1-x} = \sin(x)\displaystyle\sum_{n=0}^{\infty} x^n = \sin(x)(1 + x + x^2 + x^3 + \ldots + x^n + \ldots)$

(still converges over (-1 , 1))

(5) Differentiate a series termwise.
(No change in radius of convergence)

Given $f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots$
with radius of convergence r

$f'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1} = a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^{n-1} + \ldots$

(only if first term in original series is <u>constant</u>)

Example:

Consider $f(x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)^2} = x + \frac{x^2}{4} + \frac{x^3}{9} + \ldots + \frac{x^{n+1}}{(n+1)^2} + \ldots$

(radius of convergence 1)

$f'(x) = \sum_{n=0}^{\infty} \frac{x^n}{n+1} = 1 + \frac{x}{2} + \frac{x^2}{3} + \ldots + \frac{x^n}{n+1} + \ldots$

(radius of convergence still 1)

(6) Integrate a series termwise.
(No change in radius of convergence)

If $f(t) = \sum_{n=0}^{\infty} a_n t^n = a_0 + a_1t + a_2t^2 + \ldots + a_nt^n + \ldots$

(radius of convergence r)

then $\int_0^x f(t)\, dt = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1} = a_0x + \frac{a_1 x^2}{2} + \frac{a_2 x^3}{3} + \ldots$

(radius of convergence still r)

Example: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$

(converges everywhere $(-\infty, \infty)$)

$\int_0^x e^t\, dt = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{n+1}}{(n+1)!} \ldots$

$= e^x - 1$

(converges everywhere $(-\infty, \infty)$)

(7) Add or subtract two series with same radius of convergence.
(No change in radius of convergence)

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)\, x^n$$

Example: $$\ln(1 + x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

(integrate the series for 1/(1+x); radius of convergence is 1)

$$\ln(1 - x) = \sum_{n=0}^{\infty} (-1) \frac{x^{n+1}}{n+1} = - x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots$$

(replace "x" by "-x" in the series for ln(1+x); radius of convergence is 1)

So, $$\ln\frac{(1 + x)}{(1 - x)} = \ln(1 + x) - \ln(1 - x) = \sum_{n=0}^{\infty} ((-1)^n + 1) \frac{x^{n+1}}{n+1}$$

(radius of convergence is still 1)

(8) Product of two series (with same radius of convergence)
(No effect on radius of convergence)

$$f(x) \cdot g(x) = \sum_{n=0}^{\infty} c_n\, x^n \quad \text{where} \quad c_n = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_i\, b_j$$

Here it's best to multiply out the open forms, using the generalized distributive property, like two infinite polynomials, to obtain the first several terms and get a feeling for the nature of the new product series

Example: $$e^x \cdot \tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \cdot \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right) \cdot \left(x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right)$$

$$= \left(x + x^2 + \frac{x^3}{6} - \frac{x^4}{6} + \ldots \right)$$

$$\frac{x^3}{6} = \frac{x^3}{2} - \frac{x^3}{3} \qquad -\frac{x^4}{6} = -\frac{x^4}{3} + \frac{x^4}{6}$$

One Last Example:

Take any function like $f(x) = \dfrac{a}{b - cx}$ $\qquad$ **a, b, c** $\neq$ **0**

Find a power series that represents it, and determine its radius of convergence.

Solution: **First, divide each term in the expression by b, we obtain**

$$\frac{\frac{a}{b}}{1 - \frac{c}{b}\,x}$$

Now, *method 1*: think of this as a sum of a convergent geometric series with first term a/b and common ratio (c/b)x... it converges if $|x| < |b/c|$.

***method 2*: take the series for 1/(1-x), replace "x" with "(c/b)x", then multiply the series by a/b. Check the radius of convergence.**