Advanced Precalculus Learning Targets
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Text: *Functions Modeling Change, 2nd edition. Connally, Hughes-Hallett, Gleason, et al.*

This document provides a list of learning objectives for each unit of the Advanced Precalculus course. Although the objectives are provided within individual teaching units, the material in the course is taught and assessed in a cumulative manner. Hence, learning objectives in any given unit will be assessed in future units as well.

Unit #1: **Lines, Linear Regression, Quadratics, Rates of Change, Concavity, General Functions, and Inverse Functions** (Chapters 1, 2, and 8 of text)

1. Demonstrate understanding and fluency with multiple forms of linear functions and equations: slope-intercept form, point-slope form, and standard form, and understand, both intuitively and practically, why one form is preferable to another form in a given situation.
2. Demonstrate an understanding of the effect of parameter changes on the graph of a linear function.
3. Demonstrate an understanding of the correlation coefficient, r, for describing the linearity of a set of bivariate data, and interpret such understanding with appropriate attributes (positive vs. negative, strong vs. weak, linear vs. non-linear.)
4. Demonstrate and appreciate the key differences between an estimation, an approximate answer, and an exact answer.
5. Understand that making an inference about a population from a sample always involves uncertainty, with a key role of statistical methods being to estimate the size of the uncertainty.
6. Discuss the inherent dangers in using models for extrapolating beyond the range of the given data.
7. Understand and discuss (give specific examples) of how reader bias, measurement error, and display distortion can affect the interpretation of data.
8. Provide examples of functions and non-functions via words, pictures, formulas, and bivariate data sets.
9. Use proper function notation and demonstrate the importance of such on homework, classwork, board work, and assessments.
10. Demonstrate, via definition and graphs, an understanding of increasing, decreasing, and uniform functions.
11. Graph (without technical assistance) linear functions, quadratic functions, and piece-wise functions.
12. Given the graph of linear, quadratic, and piece-wise functions, provide the appropriate formula.
13. Intuitively understand the concept of *average rate of change* and discuss how this relates to the concavity of a function over a given interval.
14. Determine the domain and the range of a function by examination of the graph of the function.
15. Demonstrate understanding and fluency with quadratic functions in multiple forms: vertex form, factored form, and standard form, and understand, both intuitively and practically, why one form is preferable to another form in a given situation.
16. Demonstrate….both graphically and with words….an understanding of both the Vertical Line Test and the Horizontal Line Test, and how each relates to the invertibility (or lack thereof) of a function.
17. Both graphically and via algebraic manipulation, provide the inverse of an invertible function.
18. Provide domain restrictions, as needed, to create invertible functions from ones that are not invertible throughout their natural domain.
19. Construct and deconstruct compositions of functions and apply such processes to word problems involving economics, population change, etc.

Unit #2: Exponential and Logarithmic Functions (Chapers 3-4 of text)

1. Understand and apply the basic properties and theorems of roots and exponents.
2. Demonstrate (graphically, via formula, and with sentences) a conceptual and practical understanding of exponential growth and decay, including discussing the concavity of such functions.
3. Solve exponential equations by trial and error, graphically, algebraically (when possible), and via the use of logarithms.
4. Demonstrate (graphically, by formula, and via well-written sentences) an understanding of the differences between linear growth, exponential growth, and logarithmic growth.
5. Develop an understanding of the "compounding effect" by calculating the effective annual yield for a given investment when the interest is compounded annually, semi-annually, quarterly, monthly, daily, hourly, etc.
6. Develop an understanding of *e* as a transcendental number *(as defined, via exploration, as a limit),* and as the base of the natural logarithm, and notationally demonstrate the connection to *continuous* growth.
7. Apply the properties of logarithms to the solving of logarithmic equations and understand how these properties relate to the laws of exponents.
8. Add, subtract, multiply, divide, and simplify radical expressions.
9. Provide graphs of $f(x) = \log x$ and $f(x) = \ln x$, clearly labeling the graphs with at least five ordered pairs.
10. Determine the formula for an exponential or logarithmic function via examination of its graph.
11. Determine whether a given set of bivariate data represents a linear, quadratic, exponential, or logarithmic function, and then find the formula for the function.
12. Use exponential functions to explore and solve real-world problems in population growth, economics, medicine, and other areas of application.
13. Understand, both graphically and conceptually, how the change of base of a logarithmic function changes the shape of the graph of the function.
14. Use scatter plots and exponential regression to determine the function of "best fit" for a given bivariate data set that shows an exponential growth or decay pattern.
15. Connect an understanding of linear functions to exponential functions by showing that the log of any positive-valued exponential function is a linear function, and use this fact as a means of determining whether a given bivariate data set represents exponential growth.
16. Graphically illustrate an understanding of how each of the parameters in an exponential function impacts the shape of the graph.
17. Use sentences to illustrate an understanding of how each of the parameters in an exponential function impacts the starting point and rate of growth/decay for the function.

Unit #3: Transformations of Functions and their Graphs (Chapter 5 of text)

1. Recognize and sketch the graphs of many basic "parent" functions, including, but not limited to the following categories: linear, quadratic, cubic, sinusoidal, power, exponential, logarithmic, absolute value, and square root.
2. Determine the formula for a parent function (or a transformation of such) by examination of its graph.

4. Demonstrate a graphical understanding (without technical assistance) of the effects of parameter changes on functions and their graphs.
5. Transform the graph of a function through vertical and horizontal shifts, vertical and horizontal stretches and compressions, and reflections about the axes and other vertical and horizontal lines.
6. Understand the role of symmetry in the reflection of functions about horizontal lines, vertical lines, and the lines $y = \pm x$.
7. View quadratic functions as a "family of curves," created by one or more transformations of the parent function $y = x^2$
8. Understand the effect of a transformation upon a given ordered pair of a function as well as the function as a whole.
9. Know and apply the definitions of even and odd functions, and illustrate how symmetry is connected to the definitions.
10. Illustrate, both graphically and algebraically, how a horizontal stretch or compression of a logarithmic function is equivalent to a vertical shift of the function.
11. Illustrate, both graphically and algebraically, how a transformation of a function may or may not "preserve" the even or odd status of a function.
12. Algebraically find, and graphically illustrate, all "key features" of a function including roots, y-intercept, symmetry, and the crossing of horizontal asymptotes.

Unit #4: Trigonometry (Chapter 6 and Sections 7.1-7.4)

1. Understand the fundamental difference between an equation and an identity.
2. Demonstrate fluency and ease with the use of trigonometric ratios and use such to solve right triangle word problems.
3. Use the Pythagorean Theorem and its converse and properties of special right triangles to solve mathematical problems.
4. Understand and apply key terminology of the unit circle: quadrants, reference angles, and coterminal angles.
5. Provide a definition, using sentences as well as graphical illustrations, of a periodic function.
6. Use models (ex. Ferris wheels) to explore and demonstrate an understanding of the periodic nature of sinusoidal functions.
7. Show an understanding (via well-chosen examples) that functions other than trigonometric functions may be periodic in nature.
8. Define the sine and cosine function as points on the graph of a unit circle, and relate these definitions to the sinusoidal nature of the graph of these functions.
9. Graph (without technical assistance) all six trigonometric functions and their transformations by using horizontal/vertical shifts, and horizontal/vertical stretches, and show how these changes impact the period, midline, amplitude, and range of trigonometric functions.
10. Determine the formula of a trigonometric function via examination of its graph.
11. Convert easily and fluently between degree measure and radian measure and illustrate an understanding of radian measure as it relates to arc length.
12. Extend the Pythagorean Theorem to an understanding of all three Pythagorean Identities.
13. Derive the trigonometric identities: double-angle, half-angle, sum and difference of angles.
14. Believe, via intuitive understanding as well as definition, that the "negative-angle identities" are simply an alternate stating of the even or odd properties of all trigonometric functions.
15. Demonstrate an understanding of the "co-function identities" as being a direct result of the basics of right triangle trigonometry.

16. Know the exact values for the sine, cosine, tangent, secant, cosecant, and cotangent of all "special" angles, and apply trigonometric identities (such as the half-angle, double-angle, and sum/difference identities) to find the exact trigonometric values for other useful angles.

17. Apply composition of functions to trigonometric settings: *i.e.*: $\sin\left(Arc\tan\left(\frac{-\sqrt{3}}{3} \right) \right)$.

18. Derive and apply the formula for rewriting the sum of two sinusoidal functions with the same period as a single sinusoidal function. For example, convert $y = a_1 \sin(Bt) + a_2 \cos(Bt)\ to\ y = A\sin(Bt + \vartheta)$.
19. Use trigonometric identities to solve trigonometric equations.
20. Apply sinusoidal and other trigonometric models to solve real-world problems from physics, economics, medical science, and population predicting.
21. Demonstrate an understanding of inverse trigonometric functions and the need for restricted domains.
22. Graph (without technical assistance) the functions $y = Sin^{-1}(x)$, $y = Cos^{-1}(x)$, and $y = Tan^{-1}(x)$, as well as transformations of these functions.
23. Use inverse trigonometric functions to find exact and approximate solutions to trigonometric equations.
24. Find exact solutions (without technical assistance) to equations such as $Cos^{-1}(2x) = Sin^{-1}(x)$, and illustrate your answer(s) with a detailed graph.
25. Connect an understanding of exponential functions to an understanding of sinusoidal functions so as to provide <u>very</u> detailed graphs (without technical assistance) of functions such as the following:
$y = Sin(e^x)$, $y = e^x \cdot Cos(x)$, $y = \cos\left(\frac{1}{x} \right)$, $y = 2^x \cdot \sin(2x)$, $y = x + \sin(x)$, $y = x \cdot \cos(x)$

Unit #5: Polar Coordinates, Complex Numbers, and Polar Graphing (Sections 7.5, 7.6, and supplemental teaching materials)

1. Convert easily (without technical assistance) between Cartesian (rectangular) and polar coordinates, and recognize that certain applications, such as clock problems, can be worked with much more easily via the use of polar coordinates.
2. Understand that a given Cartesian ordered pair has an infinite number of polar ordered pair representations; find additional ordered pairs as needed for a given problem situation.
3. Add, subtract, multiply, and divide complex numbers in both rectangular form and in polar form, via the use of Euler's formula.
4. Use DeMoivre's Theorem to find all roots of complex numbers.
5. Determine both rectangular and polar forms for a given complex number and convert easily between the two forms.
6. Graph polar equations by hand, as well as with technology.
7. Determine the equation for a polar curve by examining its graph.
8. Draw and use an "auxiliary rectangular graph" to assist in the creation of a polar graph. For example: $r = |8\sin(2\theta)| + 4$
9. Graph cardioids, limacons, roses, spirals, and unknown "mystery polar graphs" via familiarity with categories of polar graphs as well as via auxiliary graphing.
10. Express circles (centered on and not on the origin) as well as horizontal and vertical lines via polar equations.
11. Provide polar restrictions to describe rays, segments, arcs, and other finite and infinite portions of the polar plane. Likewise, given polar restrictions, provide a graph of what is being described. *For*

example, provide a graph showing that the polar restrictions $\theta = \frac{2\pi}{3}$, $|r| > 4$ *result in two open-ended rays in opposite directions with rectangular endpoints* $(-2, 2\sqrt{3})$ *and* $(2, -2\sqrt{3})$*, and with each ray having slope* $\frac{-3\sqrt{3}}{3}$.

12. Use trigonometric identities as needed (half-angle, for example) to find the "exact" rectangular ordered pair representations for given polar ordered pairs. *For example, convert* $\left(8, \frac{11\pi}{12}^{R}\right)$ *to rectangular coordinates.*

13. Recognize when polar coordinates and equations are much more "user-friendly" for a given problem situation than rectangular coordinates and equations and vice-versa.

14. Recognize that while we have the Binomial Theorem as a tool for expanding an expression of the form $(x + yi)^n$, the use of Euler's method is much more practical in raising complex numbers to higher powers.

Unit #6: Parametric Equations (Chapter 12 of text)

1. Understand the inherent *need* for parametric equations: i.e., the ability to describe the path of a moving particle on a real number plane in relation to time, and furthermore describe <u>*how*</u> that path was taken…direction of travel, speed of travel, starting point of travel, etc.
2. Convert from parametric equations to rectangular equations by eliminating the parameter, *t*.
3. For any set of parametric equations, be able (without technical assistance) to do the following:
   (a) State any restrictions on *x*, *y*, and *t*.
   (b) Find an implicit or explicit equation for the curve. (*i.e., find an equation solely in terms of x and y*)
   (c) Provide a table of ordered pairs, along with the <u>time</u> at which the particle is at that location.
   (d) Graph, labeling well.
   (e) Provide a complete sentence giving a detailed description of the motion of the particle.

(*Included should be information about the starting point, direction of travel, rate of motion, and any other pertinent information.)*

4. Recognize that some equations can easily be represented in multiple forms: polar, rectangular, and parametric, and that each form has advantages and disadvantages.
5. Given Cartesian graphs of t vs. $x(t)$ and t vs. $y(t)$, provide a single Cartesian graph illustrating the motion of the particle being described.
6. Review and extend previously-learned understanding of conic sections by expressing circles, ellipses, and hyperbolas in parametric form, and recognizing that any such parametric form can be altered to address different directions and speeds of travel. Thus, know and apply the general parametric form for ellipses, circles, and hyperbolas.
7. Use and extend your knowledge of all previously-learned functions within the context of parametric equations: For example, graph (without technical assistance) parametric equations such as the following…

(a) $\begin{array}{l} x(t) = \ln(t-1) \\ y(t) = 2t - 1 \end{array}$ and (b) $\begin{array}{l} x(t) = 2^t \\ y(t) = 4^t \end{array}$

8. Use the process of completing the square to convert circles, ellipses, and hyperbolas from their general implicit form to standard form and then, as needed for a given problem, into parametric form.

9. Succinctly state and apply the *definition* of a circle, an ellipse, and an hyperbola.
10. Use the definition of a circle, an ellipse, or an hyperbola, to derive its standard implicit form.
11. Use piece-wise parametric equations, as needed, to describe the path of a moving particle.
12. Determine intersection points of conics algebraically and illustrate graphically, using symmetry as a helpful aid. For example, find the exact values of all intersection points of the conics $x^2 + y^2 = \frac{1}{2}$ and $x^2 - y^2 = \frac{1}{4}$.

Unit #7: Power, Polynomial, Rational, and Logistic Functions (Chapter 9 of text, and supplementary teaching materials)

1. Recognize that inverse proportionality as well as direct proportionality are special cases of power functions.
2. Discuss graphical differences between power functions and exponential functions.
3. Clearly state how the parameters of a power function affect its graph.
4. Find the formula for a power function, given two ordered pairs that lie on its graph.
5. Recognize that a polynomial function can be defined as a sum of power functions whose exponents are nonnegative integers.
6. Know the general formula for a polynomial function, and discuss the graphical implications of the leading coefficient, the degree, and the constant term.
7. Understand that a polynomial function of degree *n* can have at most *n* distinct roots, and understand how the multiplicity of a root affects the shape of the graph.
8. Demonstrate that when viewed on a large enough scale, the graph of a polynomial function can be approximated by the graph of a power function (the polynomial's leading term).
9. Use the Rational Root Theorem and the Remainder Theorem, together with synthetic division, to determine all possible rational roots of a polynomial, and use this information to provide a detailed graph.
10. Find the formula for a polynomial function via examination of its graph.
11. Provide detailed graphs of rational functions, including information about vertical and horizontal asymptotes (if any), roots (if any), the vertical intercept (if it exists), and correct symmetry, if indeed any symmetry exists.
12. Determine the long-range behavior of a rational function function by examining the ratio of the leading terms of the numerator and denominator.
13. Find the points (if any exist) at which the graph of a rational function crosses a horizontal asymptote of the function.
14. Determine the ordered pair location of any "holes" in the graph of a rational function.
15. Find a formula for a rational function via examination of its graph.
16. Describe the graphical differences between power, exponential, and logarithmic functions, and recognize that any exponential function will always (eventually) "take over" any power function.
17. Use technology to fit exponential and polynomial functions to given sets of data, recognizing the limitations of any such model.
18. Apply optimization techniques to polynomial and rational functions in order to solve real-world problems. As always, the solution must include the function that is to be maximized or minimized, a detailed graph, and a well-written concluding sentence in the context of the given problem.
19. Describe the key features of a logistic growth model and provide the formula for a logistic function, clearly stating how changes in each parameter affect the graph of the function.

20. Algebraically demonstrate how a vertical shift of a power function can also be expressed as a rational function.

Unit #8: Vectors (Chapter 10)

1. Understand the inherent difference between a scalar and a vector, and recognize that vectors are both necessary and useful in describing any physical phenomena that is described via direction as well as intensity.
2. Know, understand, and apply properties of vector addition and scalar multiplication.
3. Understand and appropriately apply knowledge of the zero vector and of unit vectors.
4. Use vectors to solve application problems involving clocks and time-pieces.
5. Use vector notation to describe and draw motion in a two-dimensional plane as well as in three-dimensional space, and to solve corresponding application problems. (*airplane problems, wind resistance, etc.)*
6. Resolve any vector into its components, and use such components to add and subtract vectors, and determine the magnitude and direction of a vector.
7. Connect knowledge of two-dimensional vectors to previously-obtained knowledge of plane geometry. (i.e., express the sides of a parallelogram via vectors, assuming the lengths of the sides and the interior angles of the parallelogram are known.)
8. Use vectors to review and alternately express the geometric properties of parallelograms, rhombuses, rectangles, trapezoids, etc.
9. If the vertices of a triangle are known, use vectors (*not* slopes) to determine whether a given angle of the triangle is right, acute, or obtuse.
10. Connect knowledge of three-dimensional vectors to previously-obtained knowledge of geometric solids. (i.e., use vectors to find the angle of intersection between diagonals of the solid.)
11. Recognize that vectors in *n*-dimensions (where *n* is an integer greater than three) are often used as "storage locations" for large data sets as a means of easily and effectively communicating information about given variables of interest.
12. Use the dot product to determine the angle between any two vectors (tail to tail) in either two or three dimensions.
13. Know and apply the properties of the dot product as an operator.
14. Use the equation $\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos\theta$ to determine whether the two vectors in question are parallel, anti-parallel, perpendicular, or none of these.
15. As needed, provide alternate solutions to vector word problems via the use of the Law of Cosines and the Law of Sines.
16. Recognize that any matrix can be viewed as a collection of row vectors and column vectors.
17. Know and apply properties of scalar multiplication and matrix arithmetic.
18. Multiply, as needed, vectors by matrices, to solve application problems.
19. Use matrix equations to solve systems of linear (and sometimes nonlinear) equations.

Unit #9: Sequences and Series (Chapter 11)

1. For a given numerical sequence, find a formula for a_n, the *n*th term of the sequence, if such a formula exists and can be found by algebraic methods.
2. Use non-numerical sequences (i.e., *J, F, M, A,…)* to assist in the development of pattern recognition.

3. Recognize and intuitively understand that any infinite sequence can be thought of as a function whose domain is the set of all positive integers.
4. Use sentences to describe the differences between arithmetic and geometric sequences, and know how to apply the formulas for the *n*th term of each of these sequence types.
5. Know (and know why) the terms *sequence* and *series* are not interchangeable.
6. For a given series, use sigma notation (if possible) to provide its "closed form."
7. Find the sum of a finite arithmetic series and the sum of both finite and infinite geometric series, if such sums exist.
8. Determine whether a given sequence converges or diverges and determine the limit, if it exists. Use "long-term behavior" knowledge from previous chapters to assist in this process.
9. Derive the formulas for the sum of the first *n* terms of arithmetic and geometric sequences.
10. Appreciate the inherent connection between linear functions and arithmetic sequences and also between exponential functions and geometric sequences.
11. Apply the process of "finite differences" to a given sequence so as to set up and solve a matrix equation for finding a formula for the *n*th term of the sequence.
12. Use sequences and series to solve real-world problems in medicine, economics, and population growth/decay.

Unit #10: Symbolic Logic and Mathematical Induction (taught from supplementary materials)

1. Understand Symbolic Logic as a two-valued logic system in which any statement is either true or false. (*Note that other types of logic systems exist.)*
2. Know that compound statements within symbolic logic consist of two or more "simple statements" with at least one connective, and understand the relative "strength" of the connectives in relation to each other.
3. Know and apply the basic truth tables for a disjunction, a conjunction, a conditional, and a biconditional.
4. Create and complete a truth table to determine if a compound statement is a contradiction, a tautology, or is indeterminate.
5. Know how to negate and then simplify compound statements.
6. Symbolize the English sentences of an "argument" so as to then apply the roughly twenty valid reasons (*modus ponens, modus tollens, conditional disjunction, disjunctive simplification, etc)* in the completion of a direct, indirect, or conditional symbolic proof.
7. Understand and apply the basics of *quantificational logic*, which includes the symbolism, meaning, use, and negation of quantified sentences of one or more variables.
8. Use Venn Diagrams as a means of illustrating and proving some tautologies.
9. Use the process of *Mathematical Induction* to establish that a given equation is true for all positive integers.

Unit #11: Step Functions, Limits, and Continuity (taught from supplementary materials)

1. State, understand and apply the definition of the greatest integer function.
2. Provide a detailed graph of the greatest integer function and transformations of it.
3. Know and apply the properties of limits. *(Limit of a Sum, limit of a product, etc.)*
4. Evaluate a limit using a variety of methods: direct substitution, graphing, algebraic manipulation, use of the conjugate of the numerator or the denominator, etc.

5. Explain the distinctions between one-sided limits, two-sided limits, and limits at infinity.
6. Understand an intuitive definition of “continuity at a point” and illustrate this understanding with the three criteria necessary for a limit to exist at a point.
7. State the various types of discontinuities and provide examples (via function and graph) illustrating each.
8. Appreciate the absolute need for standard correct notation as the groundwork for later successful study in calculus of limits, derivatives, and integrals.