

## ALTERNATING SERIES

An *Alternating Series* has the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \dots$$

or

$$\sum_{n=1}^{\infty} (-1)^n a_n = -a_1 + a_2 - a_3 + a_4 + \dots$$

**note:** terms alternate positive..negative or vice-versa  
 + - + - + or - + - + - one-by-one  
 (all  $a_n$ 's are positive)  
 Other patterns like + + - - + - - + - -  
 are not considered alternating.

**EXAMPLES:**

(1) **Alternating Harmonic Series**

$$\sum_{n=1}^{\infty} (-1)^{n-1} (1/n) = 1 - 1/2 + 1/3 - 1/4 + 1/5 \dots$$

(2)  $\sum_{n=1}^{\infty} (-1)^{n+1} (1/n!) = 1 - 1/2! + 1/3! - 1/4! + 1/5! \dots$

(3)  $\sum_{n=1}^{\infty} (-1)^{n-1} (1/2n^3) = 1/8 - 1/64 + 1/216 - 1/512 \dots$

## ALTERNATING SERIES TEST (AST) for convergence

Given an Alternating Series like  $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ ,

if the term sequence  $\begin{cases} \text{has limit } 0 \text{ (} \lim_{n \rightarrow \infty} a_n = 0 \text{)} \\ \text{and} \\ \text{is strictly decreasing (} 0 << a_{n+1} << a_n \text{)} \end{cases}$

then the series **CONVERGES**

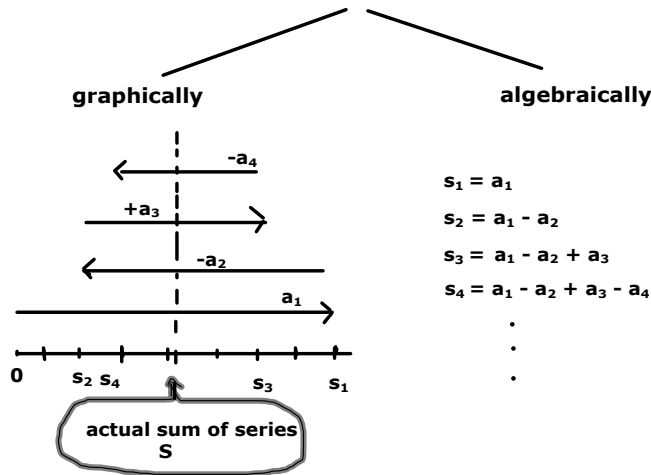
**Note:** The AST tests for convergence, not divergence  
 Of course, if  $\lim_{n \rightarrow \infty} a_n \neq 0$ , then the series diverges  
 by the  $n^{\text{th}}$  term test

**But...** if  $\lim_{n \rightarrow \infty} a_n = 0$ , and the term sequence does not strictly decrease, then you can't really conclude anything... the series could converge or it could diverge

**Nature of convergence for convergent alternating series**

Suppose  $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$  is a convergent alternating series, so its terms decrease strictly to a limit of 0

Look at the sequence of partial sums:



Major Observation:  $|S - s_n| < |s_n - s_{n+1}| = a_{n+1}$

This says...the sum S of any convergent alternating series is always caught between two successive partial sums  $s_n$  and  $s_{n+1}$  whose difference is the  $n+1^{st}$  term  $a_{n+1}$

$$|S - s_n| < |s_n - s_{n+1}| = a_{n+1}$$

distance of  $n^{th}$  partial sum from sum of series S      distance between any two successive partial sums       $n+1^{st}$  term

This also says: if you use  $s_n$  (the  $n^{th}$  partial sum) to approximate S (the sum of the entire series), your error will always be strictly less than the  $n+1^{st}$  term!

*(We use this fact for making approximations to the sum of convergent alternating series, and error-bounding our approximations...i.e. knowing within certain bounds how close our estimate is to the actual sum of the series)*

**PROBLEM 1:**

Show that the alternating harmonic series converges

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \quad \text{converges by AST}$$

*Note: This series does **not** converge absolutely*

$$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{1}{n} \right| \quad \text{is really} \quad \sum_{n=1}^{\infty} \frac{1}{n}$$

*divergent harmonic series*

Q1: In  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$

how well does  $s_{100}$  approximate the actual sum of the series?

Q2: In  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$

how many terms ( $n$ ) must you combine to be sure  $s_n$  is within 0.001 of the actual sum of the series?

**Problem 2:** Show that  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n!}$  converges

then approximate its sum with two decimal place accuracy (i.e. within  $5 \times 10^{-3}$ , or 0.005)

**Problem 3:** Approximate  $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n)^3}$

with 3 decimal place accuracy (within  $5 \times 10^{-4}$ , or 0.0005)

*(when you are finished, compare your result with  $s_{1000}$ , which is 0.112692835, as though that is the sum of the series)*