Calculus Learning Objectives

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Text: Calculus-Single and Multivariable, 3rd edition. Hughes-Hallett, Gleason, McCallum, et al.

Following is a list of general goal areas and more particular objectives within each goal area that I expect my calculus students to be able to do. This list is published as a courtesy to all interested learners, and is not intended for use outside of this specific course.

I. **Introductory Material-Function Review, Limits and Continuity**

A. Operate fluently with elementary functions.

1. Graph---without technological assistance---basic polynomials, rational functions, absolute value, exponential, trigonometric & inverse trigonometric, root, logarithmic, polar and parametric functions.
2. Graph---without technological assistance---transformations (shifts, stretches, reflections) of the functions listed above.
3. Construct and deconstruct compositions of the above functions for analysis and graphing purposes.
4. Intuitively understand and express the equation of a line in point-slope form given a point and a slope (or two points).

B. Demonstrate an appreciation for and an understanding of the notation and concept of the limit.

1. Express in words the definition of a limit of a function of a single real variable as the independent variable approaches a finite value.
2. Discuss in a mathematically correct and coherent fashion the notion of a limit (nearness idea).
3. Use the correct notation to accurately reflect the limiting behavior of a graph.
4. Believe that all major calculus ideas rest on the acceptance of the concept of the limit.
5. Have a familiarity with the epsilon-delta definition of the limit of a single variable function (BC).

C. Calculate the value of a limit where one exists.

1. Apply basic algebraic tools such as factoring, simplifying, and conjugate multiplication to determine the value of a limit (both indeterminate and determinate).
2. Use technological tools to numerically and graphically determine the value of a limit.
3. Understand and express the relationship between right and left-handed limits and the full limit.
4. Apply limit properties to determine the value of a limit.
5. Understand and verbally express why a limit may fail to exist.

D. Demonstrate an understanding of the idea of continuity of a function of a single real variable.

1. Write out the definition of continuity of a function at a point in its domain using correct limit notation.
2. Apply the definition of continuity to a variety of functions and varying point names (demonstrating flexibility with the concept of continuity).
3. List three ways in which a function can fail to be continuous at a point.
4. Extend the definition of continuity of a function at a point to an open subset of the real numbers.
5. Apply the Intermediate Value Theorem and the Extreme Value Theorem to show the existence of a function value and/or a maximum or minimum for a continuous function.

II. Beginning Calculus-History, Thoughts and Introduction to the Derivative

A. Appreciate the names and beings behind the calculus.
   1. Develop an understanding of the life of Wilhelm Gottfried Leibniz.
   2. Have knowledge of the works and tribulations of Sir Isaac Newton.
   3. Explain the conflict that resulted due to separate discoveries of calculus.
   4. Recognize and appreciate the different notations that were developed by each discoverer.

B. Exercise some “mathematical morals”.
   1. Recognize the difference between intuitive and mathematical thought processes and be able to answer questions accordingly.
   2. Realize that mathematics helps model real world phenomenon and failure to understand some mathematics therefore restricts our understanding of the world.
   3. Rephrase mathematical theorems in English words to assist comprehension and therefore application.
   4. Reflect on results to check compatibility with the way the world really works.
   5. Understand that there is a reason behind everything in mathematics and knowing the reasons helps build problem-solving power.
   6. Present a clear and plausible mathematical argument using definitions.

C. Define the derivative of a function of a single real variable.
   1. Mentally transition from the finite to the infinite, from the average rate of change to the instantaneous rate of change.
   2. Have multiple expressions (words, mathematical limit notation, graphs) to express the meaning of the derivative.
   3. Given the graph of a function, be able to sketch the graph of the derivative.
   4. Given the graph of a derivative, be able to sketch the graph of the original function.
   5. Apply the limit definition of the derivative to determine the derivative of elementary functions.
   6. Explain (mathematically and verbally) why a function may fail to be differentiable at a point.
   7. Write the equation of a tangent line to a function at a point.
   8. Develop a familiarity with and application of the Mean Value Theorem and its premises.
   9. Begin to visualize the derivative of the derivative (the second derivative, f”) and its implications on the graph of the original function.
   10. Interpret the meaning and the sign of the derivative and the second derivative in a practical context with correct units.
   11. Given a position function for a particle in rectilinear motion, analyze the motion with respect to position, velocity, speed, change in direction and acceleration.
   12. Understand the concept of linear approximation enough to resolve basic questions involving sets of discrete data and discuss the nature of the resulting estimate.
   13. Explain the relationship between differentiability and continuity.
   14. Derive the formula for the derivative of a polynomial.

III. The Definite Integral- Connecting Ideas with the Fundamental Theorem of Calculus

A. Define the definite integral of a function of a single real variable.

1. Demonstrate an understanding of the definite integral as a process of (infinite) summation.
2. Realize when Riemann sums are underestimates vs. overestimates and provide a visual and numerical estimate of the error.
3. Refine the error to a specified value.
4. Evaluate Riemann integrals as the common limit of the upper and lower Riemann sums.
5. Use proper notation for mathematically defining and expressing the definite integral.
6. Interpret the meaning of the definite integral in a practical context with correct units.
7. Have multiple understandings of what integral represents-net area, total change in the antiderivative, an average function value (when divided by the interval length).
8. Using the definition of the definite integral, surmise some of the properties of the definite integral.

B. Infer (before proving) the connection between derivatives and integrals via the Fundamental Theorem of Calculus.
1. Understand why, both mathematically and in practical contexts, the definite integral is the "inverse" of the derivative.
2. Use this fundamental idea to determine the definite integral of a rate of change function, or the derivative of an integral, both algebraically and graphically.

IV. Short-Cuts to Differentiation-The formulas

A. Intuit the derivative of an exponential function and be able to deduce the exact formula utilizing technology as an exploratory tool.
B. Apply the power rule to determine the derivative of a power function.
C. Be able to prove and apply the product and quotient rules to determine the derivative of a product or quotient of two functions.
D. Develop the derivative formula for the composition of two functions (the chain rule) and use it to determine the derivative of the composition of multiple functions.
E. Be able to state the key rules for applying rules of differentiation in words.
F. Develop the derivative formulas for the six basic trigonometric function.
G. Carry out the process of implicit differentiation to determine the derivative of an implicitly defined function.
H. Use implicit differentiation to determine the derivative of three inverse trigonometric functions (arcsin(x), arccos(x), and arctan(x).)
I. Use implicit differentiation to determine the formula for the derivative of a function's inverse.
J. Understand the difference between the quotient rule to determine the rate of change of a quotient function and L'Hopital's Rule to determine the indeterminate limit of a quotient by taking the respective derivatives (numerator and denominator).
K. Determine the derivative of a parametrically defined function (BC).
L. Interpret and use various notations for the derivative, including prime notation, differential operator notation, and Leibniz notation.
M. Use technology (namely a TI-89) to verify and discover differentiation rules.

V. Applications of the Derivative-Linear Approximation, Optimization, Relate Rates, Newton's Method for Root-Finding, and Taylor Polynomials

A. Understand that most functions are approximately linear when viewed "close-up".
1. Use the derivative to write the slope of the tangent line to a graph at a point.

2. Use local linearization (linear approximation) to approximate a curve in the vicinity of a point using a tangent line.
3. Use tangent line approximation to extrapolate and interpolate function values for a discretely defined function.

B. Realize that the derivative is useful in determining critical points (points where the function graph has a horizontal tangent or a slope which is undefined).
   1. Use the critical points and the 1^{st} and 2^{nd} derivative tests to determine places where the function has local extrema.
      a. Visualize the idea that if the first derivative changes sign about a critical a critical point, then there is a local extrema there *(First Derivative Test)*.
      b. Visualize that if the second derivative is positive or negative at a critical

         point, then the curve is concave up or concave down there, indicating a local extrema (*Second Derivative Test).*
      c. Extend this process of determining local extrema to determining points of inflection.
         i. Visualize the idea that if the *second* derivative changes sign about a critical point for the first derivative, then there is a point of inflection there.
         ii. Visualize the idea that if the third derivative is positive or negative at a critical point for the first derivative, then there is a point of inflection there.
   2. Solve practical optimization problems in business, science, the arts and social sciences.
   3. Determine global or absolute extrema by comparing local extrema to endpoint values.

C. Realize the derivative represents a rate and that this rate may be a function of other rates.
   1. Use implicit differentiation, sometimes introducing a third variable into the

      the equation, to connect several rates.
   2. Use this connection to solve for an unknown yet related rate.

D. Use the derivative in conjunction with determining roots of successive tangents in order to approximate the root of a function to a prescribed decimal place accuracy (Newton's Method of root-finding).

E. Use the first, second, third, etc derivatives of a function to build a Taylor polynomial and understand that the original function is closely modeled by this polynomial in the vicinity of its center.

VI. Constructing Antiderivatives-Graphically, Numerically, and Algebraically

A. Relate the graphs of derivatives and antiderivatives.
   1. Given the graph of the derivative, interpret function values as slopes of f with the objective of graphing ***f***.
   2. Given the graph of function ***f***, sweep the net area between it and the x-axis with the objective of graphing the antiderivative of ***f***.

B. Use the Fundamental Theorem of Calculus to algebraically construct the general antiderivative of ***f***, given an elementary function ***f***.

C. Use the Fundamental Theorem of Calculus and a technological tool to determine a specific function value given its general antiderivative and an initial condition (point).

D. Compute the total change in the antiderivative by evaluating the definite integral.

E. Apply the properties of antiderivatives.
   1. The integral of a sum is the sum of the integrals.
   2. The integral of the product of a constant and ***f*** is the product of the constant and the integral of ***f***.

F. Understand that finding the general solution to a differential equation means finding the general antiderivative.

G. Understand that finding the specific solution to a differential equation means determining the antiderivative which satisfies the differential equation and the given boundary value (initial condition).

H. Create a new *function* ***F(x)*** by integrating ***f*** from *a* to *x (Second Fundamental Theorem).*

I. Become familiar with the interpretation and manual creation of basic slope fields as a means of visualizing *all* solutions to a differential equation.

J. Be able to create a slope field on the TI-89 graphing calculator.

VII. Integration Techniques-Using Manual Methods to Determine the Antiderivative

A. Use the chain rule backwards to develop and apply the most common technique of Substitution when integrating an expression involving a function and its derivative.

B. Use the product rule backwards to develop and apply the technique of Integration by Parts when integrating a product of dissimilar functions.(BC)

C. Recognize when and how to use Trigonometric Identities to replace functions in order to apply the technique of Substitution.

D. Construct triangles to assist in the replacement of Pythagorean expressions with trigonometric expressions (in order to apply the techniques of Trigonometric Identities and Substitution).(BC)

E. Apply algebraic algorithms to decompose factorable (denominator) rational functions in order to use known antidifferentiation formulas (*Partial Fraction Decomposition).* (BC)

F. Recognize an improper integral and express it as the limit of a proper integral before evaluating that limit.

G. Utilize the TI-89 Computer Algebra System (Computer Algebra System) to reconcile manual results.

Semester I Exam (typically)

VII. Series-Types and Tests for Convergence (BC)

A. Understand what an infinite series is and what it means for a series to converge.

B. Apply the convergence properties of series when applicable.

C. Recognize when and how to apply some tests for convergence of infinite series.
   1. Test the limit of the n^{th} term as n goes to infinity and intuitively understand that if this limit is not zero, the series diverges (*n^{th} term test*).
   2. Recognize a geometric series and know when and how to apply the formula to determine its infinite sum.
   3. Know the conditions of a function ***f*** in order to associate the result of the improper integral of ***f*** to the related infinite series (*Integral Test).*
      a. Recognize the p-series and know under what conditions it diverges.
      b. Recognize the common harmonic series and know that it diverges.
      c. Recognize the alternating harmonic series and know that it converges.
   4. Compare two series' terms (one with known convergence or divergence) in order to determine dominance and thus behavior of the unknown series *(Order Comparison Test).*
   5. Be able to construct and evaluate the limit of the ratio of the $(n+1)^{st}$ term

to the n^{th} term of a series; compare this limit to 1 and be able to apply the conclusions of the *Ratio Test* for determining series convergence/divergence.

6. Know what an alternating series is and the conditions on it for convergence *(Alternating Series Test).*
    a. Be able to reconstruct the geometric reason why the error bound on the n^{th} partial sum of an alternating series is the $(n+1)^{st}$ term.
    b. Be able to apply the error bound on an alternating series to determine the sum to a specified accuracy.

D. Recognize power series and be able to apply tests (typically the Ratio Test) to determine intervals of convergence.
   1. Know that a power series always converges at its center.
   2. Know that a power series converges at points in its radius of convergence.
   3. Remember and be able to test for convergence at the endpoints of the radius of convergence.

E. Revisit the construction of the Taylor Polynomial and extend this idea to infinite series while also considering error.
   1. Develop the formula for the LaGrange Error bound, and comprehend the concept of this error bound in both a graphical and numerical sense.
   2. Apply the LaGrange Error bound to estimate the error resulting when a Taylor Series is used to approximate a function at a point.
   3. Be able to use the TI-89 *Taylor* command to create a Taylor polynomial.

F. Believe that equal to the value of any numerical estimate is the value of the error involved.

VIII. Numerical Integration-Determining the Value of a Definite Integral when Algebraic Methods Fail.

A. Review/revisit Left Hand and Right Hand Riemann sums (rectangles) for approximating the value of a definite integral.
   1. Compute these sums using the *sum seq* commands on the TI- 89 graphing calculator.
   2. Know the nature of the estimate resulting from each method (over/under) and how it depends on monotonicity of the graph.

B. Use trapezoids to approximate the value of the definite integral (*Trapezoidal Rule).*
   1. Know the nature of the resulting estimate and how it depends on the concavity of the graph.
   2. Know that estimates using the Trapezoidal Rule are computed by simply averaging the left and right hand sums.

C. Use rectangles constructed at the midpoint of each subinterval to approximate the value of the definite integral (*Midpoint Rule).*
   1. Know the nature of the resulting estimate and how it, like the Trapezoidal Rule, depends on the concavity of the graph.
   2. Know that the error resulting while using the Midpoint Rule to determine a definite integral is opposite in sign to that of the Trapezoidal Rule.
   3. Be able to manually use the midpoint rule for a small number of subdivisions ($n<10$) to determine an approximation for the definite integral.

D. Through careful comparison and error analysis of each method, create a hybrid of these methods, thus reducing the error even more (*Simpson's Rule).* (BC)

E. For each of the methods, know how the error decreases as the number of subdivisions increases by a factor of ***k***.

F. Be able to run a TI-89 program to compute resulting approximations from all methods simultaneously.

IX. Applications of Integration-Net areas, Population Densities, Volumes of Revolution, Volumes by Slicing, and Arclength

A. Subdivide a region bounded by several graphs *horizontally*, draw a typical rectangle, determine its height by subtracting (right y – left y), multiply by Δy sum up all rectangle areas, take limit as number of rectangles increases without bound, get integral and determine it to find net area of the bounded region.

B. Apply concept of integration to more unique settings such as radially decreasing population densities to determine total population.

C. Use the integral to compute volumes of solids when rectangles perpendicular and adjacent to the axis of revolution are revolved about that axis (*Volumes by Disks).*

D. Use the integral to compute volumes of solids when rectangles perpendicular and not adjacent to the axis of revolution are revolved about that axis *(Volumes by Washers).*

E. Use the integral to compute volumes of solids when rectangles parallel to the axis of revolution are revolved about that axis (*Volumes by Shells).*

F. Use the integral to compute curve length of a graph over an interval (*Arclength*).

G. Have a visceral feeling for what it means to integrate in any setting: subdivide, draw, determine the requested value for one subdivision, sum ***n*** of them, take the limit of the sum as $n \to \infty$, determine integral.

X. Differential Equations-Where the Derivative is a Function of Two Variables-Still Solving Using Graphical, Numerical and Analytic Methods

A. Verify solutions to differential equations by differentiating the solution and reconciling it with the given conditions and original differential equation.

B. Manually construct and match slope fields to their corresponding differential equations.
   1. Recognize when the slope field is a function of x, y, or both x and y.
   2. Be able to sketch the particular solution to a differential equation through a given point, using the lines of the slope field as signposts.
   3. Develop a sense for what all the solution curves look like by viewing the slope field.

C. Compute approximate points on the solution curve to a differential equation using Euler's Method. (BC)
   1. Be able to use Euler's Method programs on a TI-89 or compute values manually using a table.
   2. As for all numerical methods, realize that the question of overestimate or underestimate is imminent.
   3. Know the relationship between the number of steps used in Euler's Method and the error.

D. When the variables are separable, apply algebra to group like terms in preparation to integrate both sides (*Separation of Variables).*
   1. Recognize the solution to the differential equation dy/dx=ky to be an exponential function and verify the analytic solution using separation of variables.
   2. Apply separation of variables to solve practical problems such as those involving heating and cooling (*Newton's Laws).*
   3. Know the meaning of an equilibrium solution, when it's stable and when it's unstable.

XI. Student Groups Present Lessons-Emphasis on Textbook Learning, Content, and Presentation Skills.

A. Demonstrate how to solve the differential equation via separation of variables and partial fraction decomposition to derive the logistic function; show how to apply the logistic function to answer questions regarding appropriate situations. (BC)

B. Review the graphs of Polar Functions and teach how to differentiate and integrate them. (BC)

C. Show the investigation into applying basic calculus principles to three dimensional surfaces by introducing Multiple Integrals. (Beyond BC)

D. Venture further into Multivariate Calculus by demonstrating how to take a partial derivative and what it means to do so. (Beyond BC)

XII. Preparation for the AP Exam

A. Complete assigned College Board issued free response exams (specific to AB or BC) and be prepared to share results and queries.

B. Complete assigned College Board released multiple choice questions (specific to AB or BC) and prepare to discuss solutions and questions that arise.

C. Scrupulously and methodically correct all individual work with solutions that are provided and discussed in class.

XIII. Student Groups Apply Calculus-Post AP Project Work

A. Get a hands-on feel for the breadth and depth of calculus through discovering applications. Former topics include the calculus of economics, optimizing theater design, examining fetal blood flow rates, waste management, patterns of obesity in the U.S., corporate America vs. small business (coffee industry), erosion patterns of the coastal U.S., kayak design and construction, the calculus of SIRs, the calculus of the swine flu, the increase of ADHD diagnoses, heart rates, stress and calculus, Chronic Wasting Disease, and many more!

B. Practice the life skill of good communication.

1. State the topic question.
2. Explain supporting examples and experimental design.
3. Express calculus ideas in clear English that is understandable by others.
4. Communicate effectively in groups.
5. Divide workload to achieve mission.
6. Produce a well-organized discussion.
7. Utilize current technology to effectively demonstrate concepts.