Series continued, convergence, divergence

Warm ups

(5) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} =$

$|3x| < 1$
$|x| < \frac{1}{3}$ $\quad -\frac{1}{3} < x < \frac{1}{3}$ $\quad \frac{2}{1-3x}$

(6) Find a closed form for

geometric $\quad a=2 \quad r=3x$

$2+6x+18x^2+54x^3+\ldots = \displaystyle\sum_{n=0}^{\infty} 2\cdot(3x)^n$

and determine where (for what values of x) it converges and what it converges to.

Jan 6-8:31 PM

Convergence/Divergence of a *Sequence*

A sequence $\{a_n\}$ *converges*

if and only if its *terms* get closer and closer to some real number L (limit), as $n \to \infty$

(otherwise, that is, if the terms do not settle down around some fixed real number L, the sequence *diverges*)

Formally stated:

$\lim_{n\to\infty} a_n = L$ means L is a real number,

and for any $\epsilon > 0$ there is a cutoff real number N such that if $n > N$, $|a_n - L| < \epsilon$

Jan 2-10:06 AM

Converge or Diverge?

- $\{3n/(n+2)\}_{n=1}^{\infty}$ .... C $\lim_{n\to\infty} \frac{3n}{n+2} = 3$
- $\{1/n\}_{n=1}^{\infty}$ .... C $\lim_{n\to\infty} \frac{1}{n} = 0$
- $\{(-1)^n\}_{n=0}^{\infty}$ .... D $\lim_{n\to\infty} (-1)^n$ DNE oscillates
- $\{n\}_{n=1}^{\infty}$ .... D $\lim_{n\to\infty} (n)$ DNE
- $\{\tan^{-1} n\}_{n=1}^{\infty}$ .... C $\lim_{n\to\infty} \tan^{-1} n = \frac{\pi}{2}$
- $\{(2n^2 + 5n - 7)/(5n^2 + 3n + 4)\}_{n=1}^{\infty}$ .... C $\lim_{n\to\infty} a_n = \frac{2}{5}$

Jan 2-10:29 AM

Convergence of a *Series*

A Series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots a_n + \ldots$ **Converges**

if and only if its *sequence of partial sums* converges

- **If a series $\sum_{n=1}^{\infty} a_n$ CONVERGES, then $\boxed{\lim_{n\to\infty} s_n = S}$ (a number),**

  and we call the number S that the series converges to the "sum of the series"

- **If a series $\sum_{n=1}^{\infty} a_n$ does NOT converge (i.e. $\lim_{n\to\infty} s_n$ fails to exist) DNE**

  then we say the series DIVERGES

Jan 2-10:37 AM

Examples of ***Series*** Convergence

$$\sum_{n=0}^{\infty} \frac{1}{(2^n)}$$

$\left(\frac{1}{2}\right)^n$ geometric $\quad r=\frac{1}{2} \quad \left|\frac{1}{2}\right|<1 \quad S=\frac{1}{1-\frac{1}{2}}=2$

Converges (to 2) geom.

$1,\ 1+\frac{1}{2},\ 1+\frac{1}{2}+\frac{1}{4},\ \ldots$

$S_1 \quad S_2 \quad S_3$

$$\sum_{k=1}^{\infty} k$$

Diverges $\lim_{n\to\infty} S_n$ DNE

$1,\ 1+2,\ 1+2+3,$

$S_1 \quad S_2$

$$\sum_{n=0}^{\infty} (-1)^n$$

Diverges. oscillates

Jan 2-10:41 AM

ce Properties of Series

Linearity If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ <u>both</u> converge

then (i) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$ and

(ii) $\sum_{n=1}^{\infty} ka_n = k\sum_{n=1}^{\infty} a_n$ (k being any constant)

Truncation Altering a series by a finite number of terms does NOT affect convergence or divergence (for a <u>convergent</u> series, however, it may affect the <u>sum</u> of the series)

Rearrangement or Regrouping

If a series has all <u>positive</u> terms and it converges, then rearranging or regrouping the terms has no affect on convergence...the series will still converge to the same sum

Jan 2-10:59 AM

☆☆ n^{th} term test (for divergence) ☆☆

It is not correct to assume that if $\lim_{n\to\infty} a_n = 0$ then the series $\sum_{n=0}^{\infty} a_n$ converges

Statement of theorem:

$a \to b$

(i.e. The converse is not true)

If a series $\underbrace{\sum_{n=1}^{\infty} a_n \text{ converges}}_{\text{If } a}$, then $\underbrace{\lim_{n\to\infty} a_n = 0}_{b}$

If a · then b

$\sim b \to \sim a$

More Useful Form (Contrapositive Equivalent):

Know This

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ DIVERGES

Jan 2-10:47 AM

EXAMPLES:

$\sum_{n=1}^{\infty} \tan^{-1} n$

nth term test

$\lim_{n\to\infty} \tan^{-1} n = \frac{\pi}{2} \neq 0 \therefore \sum_{n=0}^{\infty} \tan^{-1} n$ DIVERGES

$\sum_{j=5}^{\infty} 2j/(3j-7)$

nth term test

$\lim_{n\to\infty} \frac{2n}{3n-7} = \frac{2}{3} \neq 0 \therefore \sum_{j=5}^{\infty} \frac{2j}{3j-7}$ diverges

Does $\underbrace{\sum_{n=1}^{\infty} 1/n}_{\text{HARMONIC SERIES}}$

nth term

$\lim_{n\to\infty} \frac{1}{n} = 0$ so we don't know

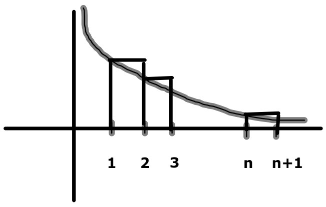
if $\sum_{n=1}^{\infty} \frac{1}{n}$ converges or diverges.

Jan 2-10:54 AM

HARMONIC SERIES IS DIVERGENT

$$\sum_{n=1}^{\infty} \frac{1}{n}$$ DIVERGES VERY SLOWLY

Reason: Look at the continuous curve f(x) = 1/x and compare the area under that curve over [1, n+1] with s_n = 1 + 1/2 + ... 1/n



$$\int_1^{n+1} 1/x \; dx = \qquad \underbrace{1 + 1/2 + 1/3 + \ldots + 1/n}_{s_n}$$

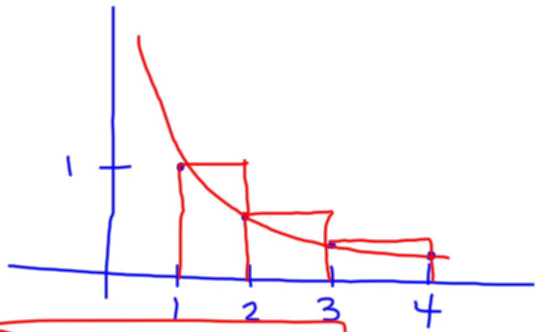
$$\lim_{n \to \infty} \ln(n+1) =$$

$$\text{so } \lim_{n \to \infty} s_n = \qquad \therefore \text{ series}$$

Jan 2-11:02 AM

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} +$$

$$\int_1^{\infty} \frac{1}{x} dx$$



$$\int_1^{\infty} \frac{1}{x} dx < \sum_{n=1}^{\infty} \frac{1}{n} \quad \boxed{\text{DIVERGES}}$$

circumscribed rect of width 1

$$\lim_{b \to \infty} \int_1^{b} \frac{1}{x} dx$$

$$\lim_{b \to \infty} \ln|x| \Big|_1^b = \lim_{b \to \infty} \ln|b| - \ln|1| \quad \text{DIVERGES!}$$

Jan 26-2:45 PM

INTEGRAL TEST

Suppose f is continuous, positive-valued, decreasing function on $[c, \infty)$

can check f' (want $f'(n) < 0$)

make sure this is the case!

then the infinite series $\sum_{n=c}^{\infty} f(n)$

does whatever the improper integral $\int_c^{\infty} f(x)\,dx$ does

that is, they both converge (not necessarily to the same number), or they both diverge

Jan 2-11:10 AM

$\sum_{n=1}^{\infty} \frac{1}{n^2}$

$\sum_{p=1}^{\infty} \frac{3}{p+4}$ diverges

$3\int_1^{\infty} \frac{1}{x+4}\,dx$ diverges

nth term test

$\lim_{n\to\infty} \frac{1}{n^2} = 0$

integral test

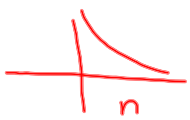
$\int_1^{\infty} \frac{1}{x^2}\,dx$ converges

$p > 1$

$\int_1^{\infty} \frac{1}{x^p}\,dx$

$\frac{1}{p-1}$

$\therefore$ converges



- $\frac{1}{x^2}$ is pos. valued
- $\frac{1}{x^2}$ is dec. since

$\rightarrow D_x \frac{1}{x^2} = -\frac{2}{x^3} < 0$ for $x > 0$ $(n > 0)$

Jan 26-2:52 PM

An important consequence of the INTEGRAL TEST

$$\boxed{\text{p - series} \quad \sum_{n=1}^{\infty} \frac{1}{n^p}}$$

$1/(n^p)$

Recall p-integrals? $\displaystyle\int_1^{\infty} \frac{1}{x^p}\,dx$ $\begin{cases} \text{converge if } p>1 \\ \text{diverge if } p \le 1 \end{cases}$

p - series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ $\begin{cases} \text{converge if } p>1 \\ \text{diverge if } p \le 1 \end{cases}$

Jan 2-11:13 AM

EXAMPLES: p-series, integral test

(1) $\displaystyle\sum_{n=5}^{\infty} \frac{1}{n^3}$ converges p-series $p = 3 > 1$

$n=5$ does not matter

(2) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ $\frac{1}{k^{1/2}}$ diverges p-series $p = \frac{1}{2} < 1$

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.001}}$ converges p-series $p = 1.001 > 1$

(4) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(n \cdot \ln(n))}$

(must use integral test)

$$\int_2^{\infty} \frac{1}{x\ln x}\,dx = \lim_{b\to\infty} \int_2^{b} \frac{1}{\cancel{x}\ln x}\,dx = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{w}\,dw$$

$w = \ln x$
$dw = \frac{1}{x}dx$

$\frac{1}{x\ln x} > 0$ for $x > 2$

$\boxed{\text{Show}}$

$$\frac{1}{(n+1)\ln(n+1)} < \frac{1}{n\ln n}$$

OR

$$D_x \frac{1}{x\ln x} < 0$$

$$D_x(x\ln x)^{-1} = -(x\ln x)^{-2} \cdot \left[x \cdot \frac{1}{x} + \ln x\right]$$

$$\underbrace{\frac{-1}{(x\ln x)^2}}_{\text{neg}} \underbrace{\left[1 + \ln x\right]}_{\text{pos.}} < 0$$

For $x \ge 2$

$$= \lim_{b\to\infty} \ln|w| \Big|_{\ln 2}^{\ln b}$$

$$= \lim_{b\to\infty} \ln|\ln b| - \ln|\ln 2|$$

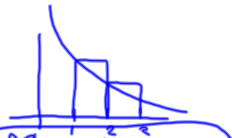
DNE (∞)

$\therefore$ Series diverges

Jan 2-11:16 AM

1) Write the n^{th} term test

2) Write the integral test



$\boxed{\sum_{n=1}^{\infty} \frac{1}{n} \text{ harmonic diverges}}$

Converge or Diverge? + WHY

3) $\sum_{p=1}^{\infty} \frac{1}{p^5}$ p-series $p=5$ converges (or $\int_1^{\infty} \frac{1}{x^5}dx$ conv)

4) $\sum_{n=1}^{\infty} \frac{4^{n-1}}{5^n}$ conv. geom $r=\frac{4}{5}<1$

$\frac{1}{4}\frac{4^n}{5^n}$

$\frac{1}{4}\left(\frac{4}{5}\right)^n$

$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$

5) $\sum_{n=1}^{\infty} \frac{(n-2)!}{n!}$

$\frac{1\cancel{(n-2)!}}{n(n-1)\cdot\cancel{(n-2)!}}$ converges

$\int_1^{\infty} \frac{1}{x(x-1)}dx$

$\int_1^{\infty} \frac{1}{x^2}dx$ converges

integral test...

p-series, $p=2$

Jan 27-2:15 PM

REVIEW

We now know:

Geometric Series $\sum_{n=0}^{\infty} a\cdot r^n$

converges if $|r|<1$ to $\frac{a}{1-r}$

p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$

Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

Alt. Harmonic $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converges

2 Tests for convergence/divergence:

- **n^{th} term test** If $\lim_{n\to\infty} a_n \neq 0$ the series $\sum_{n=1}^{\infty} a_n$ diverges
- **integral test**

Check conditions (pos. val. + decreasing $f(n+1)<f(n)$ or $f'(n)<0$)

The series does whatever the integral does.

Jan 2-11:19 AM

DO 9.2 in your text

#1,4,6,7,9,11-15,19,21,23,24,31

Jan 6-10:10 PM