Joseph Louis Lagrange (1736 - 1813)

From `A Short Account of the History of Mathematics' (4th edition, 1908) by W. W. Rouse Ball.

Joseph Louis Lagrange, the greatest mathematician of the eighteenth century, was born at Turin on January 25, 1736, and died at Paris on April 10, 1813. His father, who had charge of the Sardinian military chest, was of good social position and wealthy, but before his son grew up he had lost most of his property in speculations, and young Lagrange had to rely for his position on his own abilities. He was educated at the college of Turin, but it was not until he was seventeen that he shewed any taste for mathematics - h interest in the subject being first excited by a memoir by Halley across which he came by accident. Alone and unaided he threw himself into mathematical studies; at the end of a year's incessant toil he was already an accomplished mathematician, and was made a lecturer in the artillery school.

The first fruit of Lagrange's labours here was his letter, written when he was still only nineteen, to Euler, in which he solved the isoperimetrical problem which for more than half a century had been a subject of discussion. To effect the solution (in which he sought to determine the form of a function so that a formula in which it entered should satisfy a certain condition) he enunciated the principles of the calculus of variations. Euler recognized the generality of the method adopted, and its superiority to that used by himself; and with rare courtesy he withheld a paper he had previously written, which covered some of the same ground, in order tha the young Italian might have time to complete his work, and claim the undisputed invention of the new calculus. The name of this branch of analysis was suggested by Euler. This memoir at once placed Lagrange in the front rank of mathematicians then living.

In 1758 Lagrange established with the aid of his pupils a society, which was subsequently incorporated as the Turin Academy, and i the five volumes of its transactions, usually known as the *Miscellanea Taurinensia*, most of his early writings are to be found. Many of these are elaborate memoirs. The first volume contains a memoir on the theory of the propagation of sound; in his he indicates a mistake made by Newton, obtains the general differential equation for the motion, and integrates it for motion in a straight line. This volume also contains the complete solution of the problem of a string vibrating transversely; in this paper he points out a lack of generality in the solutions previously given by Taylor, D'Alembert, and Euler, and arrives at the conclusion that the form of the curv at any time t is given by the equation $y = a \sin mx \sin nt$. The article concludes with a masterly discussion of echoes, beats, and compound sounds. Other articles in this volume are on recurring series, probabilities, and the calculus of variations.

The second volume contains a long paper embodying the results of several memoirs in the first volume on the theory and notation o the calculus of variations; and he illustrates its use by deducing the principle of least action, and by solutions of various problems in dynamics.

The third volume includes the solution of several dynamical problems by means of the calculus of variations; some papers on the integral calculus; a solution of Fermat's problem mentioned above, to find a number x which will make $(x^2n + 1)$ a square where n i a given integer which is not a square; and the general differential equations of motion for three bodies moving under their mutual attractions.

In 1761 Lagrange stood without a rival as the foremost mathematician living; but the unceasing labour of the preceding nine years ha seriously affected his health, and the doctors refused to be responsible for his reason or life unless he would take rest and exercise. Although his health was temporarily restored his nervous system never quite recovered its tone, and henceforth he constantly suffere from attacks of profound melancholy.

The next work he produced was in 1764 on the libration of the moon, and an explanation as to why the same face was always turned to the earth, a problem which he treated by the aid of virtual work. His solution is especially interesting as containing the gern of the idea of generalized equations of motion, equations which he first formally proved in 1780.

He now started to go on a visit to London, but on the way fell ill at Paris. There he was received with marked honour, and it was with regret he left the brilliant society of that city to return to his provincial life at Turin. His further stay in Piedmont was, however, short. In 1766 Euler left Berlin, and Frederick the Great immediately wrote expressing the wish of ``the greatest king in Europe" to have ``the greatest mathematician in Europe" resident at his court. Lagrange accepted the offer and spent the next twenty years in Prussia, where he produced not only the long series of memoirs published in the Berlin and Turin transactions, but his monumental work, the *Mécanique analytique*. His residence at Berlin commenced with an unfortunate mistake. Finding most of his colleagues married, and assured by their wives that it was the only way to be happy, he married; his wife soon died, but the union was not a happy one.

Lagrange was a favourite of the king, who used frequently to discourse to him on the advantages of perfect regularity of life. The lesson went home, and thenceforth Lagrange studied his mind and body as though they were machines, and found by experiment th exact amount of work which he was able to do without breaking down. Every night he set himself a definite task for the next day, and on completing any branch of a subject he wrote a short analysis to see what points in the demonstrations or in the subject-matte were capable of improvement. He always thought out the subject of his papers before he began to compose them, and usually wrot them straight off without a single erasure or correction.

His mental activity during these twenty years was amazing. Not only did he produce his splendid *Mécanique analytique*, but he contributed between one and two hundred papers to the Academies of Berlin, Turin, and Paris. Some of these are really treatises, and all without exception are of a high order of excellence. Except for a short time when he was ill he produced on average about one memoir a month. Of these I note the following as amongst the most important.

First, his contributions to the fourth and fifth volumes, 1766-1773, of the *Miscellanea Taurinensia*; of which the most important wa the one in 1771, in which he discussed how numerous astronomical observations should be combined so as to give the most probable result. And later, his contributions to the first two volumes, 1784-1785, of the transactions of the Turin Academy; to the first of which he contributed a paper on the pressure exerted by fluids in motion, and to the second an article on integration by infinit series, and the kind of problems for which it is suitable.

Most of the memoirs sent to Paris were on astronomical questions, and among these I ought particularly to mention his memoir on the Jovian system in 1766, his essay on the problem of three bodies in 1772, his work on the secular equation of the moon in 1773 and his treatise on cometary perturbations in 1778. These were all written on subjects proposed by the French Academy, and in each case the prize was awarded to him.

The greater number of his papers during this time were, however, contributed to the Berlin Academy. Several of them deal with questions on *algebra*. In particular I may mention the following. (i) His discussion of the solution in integers of indeterminate quadratics, 1769, and generally of indeterminate equations, 1770. (ii) His tract on the theory of elimination, 1770. (iii) His memoirs on the general process for solving an algebraical equation of any degree, 1770 and 1771; this method fails for equations of an orde above the fourth, because it then involves the solution of an equation of higher dimensions than the one proposed, but it gives all the solutions of his predecessors as modifications of a single principle. (iv) The complete solution of a binomial equation of any degree; this is contained in the memoirs last mentioned. (v) Lastly, in 1773, his treatment of determinants of the second and third order, and of invariants.

Several of his early papers also deal with questions connected with the neglected but singularly fascinating subject of the *theory of numbers*. Among these are the following. (i) His proof of the theorem that every integer which is not a square can be expressed as the sum of two, three, or four integral squares, 1770. (ii) His proof of Wilson's theorem that if n be a prime, then (n-1)! + 1 is alway a multiple of n, 1771. (ii) His memoirs of 1773, 1775, and 1777, which give the demonstrations of several results enunciated by Fermat, and not previously proved. (iv) And, lastly, his method for determining the factors of numbers of the form $x^2 + ay^2$.

There are also numerous articles on various points of *analytical geometry*. In two of them, written rather later, in 1792 and 1793, he reduced the equations of the quadrics (or conicoids) to their canonical forms.

During the years from 1772 to 1785 he contributed a long series of memoirs which created the science of *differential equations*, a any rate as far as partial differential equations are concerned. I do not think that any previous writer had done anything beyond considering equations of some particular form. A large part of these results were collected in the second edition of Euler's integral calculus which was published in 1794.

Lagrange's papers on *mechanics* require no separate mention here as the results arrived at are embodied in the *Mécanique analytique* which is described below.

Lastly, there are numerous memoirs on problems in *astronomy*. Of these the most important are the following. (i) On the attraction of ellipsoids, 1773: this is founded on Maclaurin's work. (ii) On the secular equation of the moon, 1773; also noticeable for the earliest introduction of the idea of the potential. The potential of a body at any point is the sum of the mass of every element of the body when divided by its distance from the point. Lagrange shewed that if the potential of a body at an external point were known, the attraction in any direction could be at once found. The theory of the potential was elaborated in a paper sent to Berlin in 1777. (iii) On the motion of the nodes of a planet's orbit, 1774. (iv) On the stability of the planetary orbits, 1776. (v) Two memoirs in whic the method of determining the orbit of a comet from three observations is completely worked out, 1778 and 1783: this has not indeed proved practically available, but his system of calculating the perturbations by means of mechanical quadratures has formed the basis of most subsequent researches on the subject. (vi) His determination of the secular and periodic variations of the elements of the planets, 1781-1784: the upper limits assigned for these agree closedly with those obtained later by Leverrier, and Lagrange proceeded as far as the knowledge then possessed of the masses of the planets permitted. (vii) Three memoirs on the method of interpolation, 1783, 1792 and 1793: the part of finite differences dealing therewith is now in the same stage as that in which Lagrang left it.

Over and above these various papers he composed his great treatise, the *Mécanique analytique*. In this he lays down the law of virtual work, and from that one fundamental principle, by the aid of the calculus of variations, deduces the whole of mechanics, both of solids and fluids. The object of the book is to shew that the subject is implicitly included in a single principle, and to give general formulae from which any particular result can be obtained. The method of generalized co-ordinates by which he obtained this result perhaps the most brilliant result of his analysis. Instead of following the motion of each individual part of a material system, as D'Alembert and Euler had done, he shewed that, if we determine its configuration by a sufficient number of variables whose numbe is the same as that of the degrees of freedom possessed by the system, then the kinetic and potential energies of the system can be expressed in terms of those variables, and the differential equations of motion thence deduced by simple differentiation. For exampl in dynamics of a rigid system he replaces the consideration of the particular problem by the general equation, which is now usually written in the form

Amongst other minor theorems here given I may mention the proposition that the kinetic energy imparted by the given impulses to a material system under given constraints is a maximum, and the principle of least action. All the analysis is so elegant that Sir William Rowan Hamilton said the work could only be described as a scientific poem. It may be interesting to note that Lagrange remarked that mechanics was really a branch of pure mathematics analogous to a geometry of four dimensions, namely, the time and the three coordinates of the point in space; and it is said that he prided himself that from the beginning to the end of the work there was not a single diagram. At first no printer could be found who would publish the book; but Legendre at last persuaded a Paris firm to undertake it, and it was issued under his supervision in 1788.

In 1787 Frederick died, and Lagrange, who had found the climate of Berlin trying, gladly accepted the offer of Louis XVI. to migra to Paris. He received similar invitations from Spain and Naples. In France he was received with every mark of distinction, and special apartments in the Louvre were prepared for his reception. At the beginning of his residence here he was seized with an attac of the melancholy, and even the printed copy of his *Mécanique* on which he had worked for a quarter of a century lay for more tha two years unopened on his desk. Curiosity as to the results of the French revolution first stirred him out of his lethargy, a curiosity which soon turned to alarm as the revolution developed. It was about the same time, 1792, that the unaccountable sadness of his lif and his timidity moved the compassion of a young girl who insisted on marrying him, and proved a devoted wife to whom he becam warmly attached. Although the decree of October, 1793, which ordered all foreigners to leave France, specially exempted him by name, he was preparing to escape when he was offered the presidency of the commission for the reform of weights and measures. The choice of the units finally selected was largely due to him, and it was mainly owing to his influence that the decimal subdivision was accepted by the commission of 1799.

Though Lagrange had determined to escape from France while there was yet time, he was never in any danger; and the different revolutionary governments (and at a later time, Napoleon) loaded him with honours and distinctions. A striking testimony to the respect in which he was held was shown in 1796 when the French commissary in Italy was ordered to attend in full state on Lagrange's father, and tender the congratulations of the republic on the achievements of his son, who ``had done honour to all mankind by his genius, and whom it was the special glory of Piedmont to have produced." It may be added that Napoleon, when h attained power, warmly encouraged scientific studies in France, and was a liberal benefactor of them.

In 1795 Lagrange was appointed to a mathematical chair at the newly-established École normale, which enjoyed only a brief existence of four months. His lectures here were quite elementary, and contain nothing of any special importance, but they were published because the professors had to ``pledge themselves to the representatives of the people and to each other neither to read nor to repeat from memory," and the discourses were ordered to be taken down in shorthand in order to enable the deputies to se how the professors acquitted themselves.

On the establishment of the École polytechnique in 1797 Lagrange was made a professor; and his lectures there are described by mathematicians who had the good fortune to be able to attend them, as almost perfect both in form and matter. Beginning with the merest elements, he led his hearers on until, almost unknown to themselves, they were themselves extending the bounds of the subject: above all he impressed on his pupils the advantage of always using general methods expressed in a symmetrical notation.

His lectures on the differential calculus form the basis of his *Théorie des fonctions analytiques* which was published in 1797. This work is the extension of an idea contained in a paper he had sent to the Berlin Memoirs in 1772, and its object is to substitute for th differential calculus a group of theorems based on the development of algebraic functions in series. A somewhat similar method had been previously used by John Landen in the *Residual Analysis*, published in London in 1758. Lagrange believed that he could thus get rid of those difficulties, connected with the use of infinitely large and infinitely small quantities, to which philosophers objected in the usual treatment of the differential calculus. The book is divided into three parts: of these, the first treats of the general theory of functions, and gives an algebraic proof of Taylor's theorem, the validity of which is, however, open to question; the second deals wi applications to geometry; and the third with applications to mechanics. Another treatise on the same lines was his *Leçons sur le calcul des fonctions*, issued in 1804. These works may be considered as the starting-point for the researches of Cauchy, Jacobi ar Weierstrass.

At a later period Lagrange reverted to the use of infinitesimals in preference to founding the differential calculus on the study of algebraic forms; and in the preface to the second edition of the *Mécanique*, which was issued in 1811, he justifies the employment o infinitesimals, and concludes by saying that ``when we have grasped the spirit of the infinitesimal method, and have verified the exactness of its results either by the geometrical method of prime and ultimate ratios, or by the analytical method of derived function we may employ infinitely small quantities as a sure and valuable means of shortening and simplifying our proofs."

His *Résolution des équations numériques*, published in 1798, was also the fruit of his lectures at the Polytechnic. In this he gives the method of approximating to the real roots of an equation by means of continued fractions, and enunciates several other theorem In a note at the end he shows how Fermat's theorem that

where p is a prime and a is prime to p, may be applied to give the complete algebraical solution of any binomial equation. He also here explains how the equation whose roots are the squares of the differences of the roots of the original equation may be used so a to give considerable information as to the position and nature of those roots.

The theory of the planetary motions had formed the subject of some of the most remarkable of Lagrange's Berlin papers. In 1806 th subject was reopened by Poisson, who, in a paper read before the French Academy, showed that Lagrange's formulae led to certai limits for the stability of the orbits. Lagrange, who was present, now discussed the whole subject afresh, and in a memoir communicated to the Academy in 1808 explained how, by the variation of arbitrary constants, the periodical and secular inequalities of any system of mutually interacting bodies could be determined.

In 1810 Lagrange commenced a thorough revision of the *Mécanique analytique*, but he was able to complete only about two-third of it before his death.

In appearance he was of medium height, and slightly formed, with pale blue eyes and a colourless complexion. In character he was nervous and timid, he detested controversy, and to avoid it willingly allowed others to take credit for what he had himself done. Lagrange's interests were essentially those of a student of pure mathematics: he sought and obtained far-reaching abstract results, an was content to leave the applications to others. Indeed, no inconsiderable part of the discoveries of his great contemporary, Laplace consists of the application of the Lagrangian formulae to the facts of nature; for example, Laplace's conclusions on the velocity of sound and the secular acceleration of the moon are implicitly involved in Lagrange's results. The only difficulty in understanding Lagrange is that of the subject-matter and the extreme generality of his processes; but his analysis is ``as lucid and luminous as it is symmetrical and ingenious."

A recent writer speaking of Lagrange says truly that he took a prominent part in the advancement of almost every branch of pure mathematics. Like Diophantus and Fermat, he possessed a special genius for the theory of numbers, and in this subject he gave solutions of many of the problems which had been proposed by Fermat, and added some theorems of his own. He created the calculus of variations. To him, too, the theory of differential equations is indebted for its position as a science rather than a collectio of ingenious artifices for the solution of particular problems. To the calculus of finite differences he contributed the formula of interpolation which bears his name. But above all he impressed on mechanics (which it will be remembered he considered a branch of pure mathematics) that generality and completeness towards which his labours invariably tended.

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More Taylor Series
Estimating Error

What is the closed form of the Maclaurin series for

e^x?

sinx?

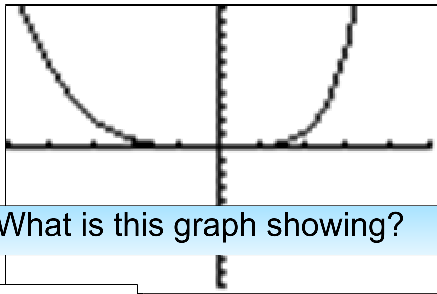
cosx?

Graph the absolute value of the difference between the actual function and the polynomial function estimate.

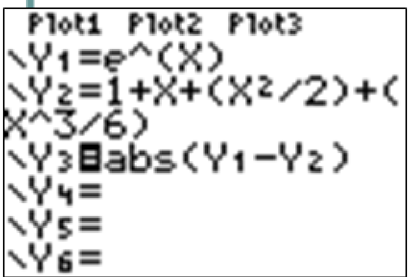
$f(x) = e^x$

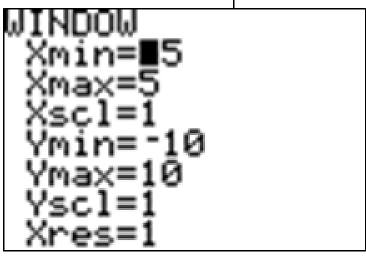
compared to

$T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$



What is this graph showing?





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What are the 2 sources of error in a Taylor/Maclaurin approximation?

- The number of terms (degree) of the polynomial (The fewer terms you have, the higher the error)

- How far the *x*-value is from the center point (The farther away from the center point, the higher the error)

  - Errors are specific to the *x*-value you are evaluating. Change the *x*-value and you need to recalculate the error.

Taylor series are used to estimate the value of functions (at least theoretically - now we can usually use the calculator or computer to calculate directly.)

An estimate is only useful if we have an idea of how accurate the estimate is.

When we use part of a Taylor series to estimate the value of a function, the end of the series that we do not use is called the remainder. If we know the size of the remainder, then we know how close our estimate is, or we can find a bound on our estimate (that is, the maximum size of error).

$$f(x) = T_n(x) + R_n(x)$$

Exact Value — $f(x)$

Approximate Value — $T_n(x)$

Remainder (error) — $R_n(x)$

→

Today's Objectives

$$f(x) = T_n(x) + R_n(x)$$

Bounding this error is our challenge

We have focused on finding the approximating polynomial.

Today we study the remainder (error).

1. Determine the maximum size of the remainder for Alternating Series
2. Find the maximum size of the remainder using Lagrange remainder

Taylor's Theorem with Remainder

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

Notice that there is an "=" sign, not an " $\approx$ " sign because now, the Remainder represents the rest of the series.

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Remainder after partial sum S_n where c is between a and x.

Lagrange Form of the Remainder

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

on

Taylor polynomial of degree n.

Looks like next term of Taylor poly except for c

c is between x (point of interest) & a (center of polynomial)

If M is the maximum value of $f^{(n+1)}(x)$ on the interval between a and x, then:

Remainder Estimation Theorem

$$\left|R_n(x)\right| \le \frac{M}{(n+1)!}\left|x-a\right|^{n+1}$$

Note that this is the formula that is in our book.

Soooooo….we will frequently be seeking to bound the n+1st derivative value at 'c'! That will be key…

(ex)

Find the error bound on a Mac. poly of degree 5 when approximating Sin(.2).

$$R_{5}(6) = \frac{f^{vii}(c)}{7!}(x-0)^7 \leq \frac{1}{7!}(.2)^7$$

Bound: $f^{vii}(c)$

bound.

What's the difference?

The value you use for the derivative.

$$R_n(x) \le \frac{f^{(n+1)}(x)}{(n+1)!}(x-a)^{n+1}$$

$$\left|R_n(x)\right| \le \frac{M}{(n+1)!}\left|(x-a)^{n+1}\right|$$

- *M* is the largest value for the (n+1) th derivative on interval [*a*,*x*]

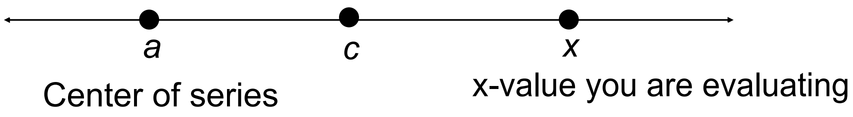
Note on Notation

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$

Is sometimes written as

$$|R_n(x)| \le \frac{f^{n+1}(c)}{(n+1)!}|x-a|^{n+1}$$

Where *c* is a value in the interval [*a*,*x*] that maximizes the value of f^{n+1}



$$\left|R_n(x)\right| \le \left|\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}\right|$$

This is called the error bound on the LaGrange Remainder. We will try to Find a positive real number E that lies above it.

If successful, then

$$\left|R_n(x)\right| < E$$

And since $f(x) = P_n(x) + R_n(x)$

Then $f(x) - P_n(x) = R_n(x)$

And $\left|f(x) - P_n(x)\right| = \left|R_n(x)\right| < E$

$$-E < f(x) - P_n(x) < E$$

<u>So the Taylor Polynomial at ‘x’ is within E of the true value f(x)</u>

When given a problem involving error bounding on a non-alternating series:

- Write the general LaGrange Remainder. Then put the specifics for the problem at hand in (n+1st derivative at c, n+1, $(x-a)^{n+1}$).

- Focus on the n+1st derivative and at what point c between the center and x it would be largest in value.

- Bound the LaGrange Remainder using the bound on the n+1st derivative at your chosen c value.

- If asked to find an estimate accurate to k decimal places, this means error must be bounded by $5 \times 10^{-(k+1)}$ (so accurate to 3 decimal places means error is less than 0.0005).

- If possible after determining error bound, determine actual error to see if your bound bounds it. Also think about the situation graphically; it will aid your understadning of what you are doing.