## 8.1 Introduction

In Chapter 6, we explored the relationship between the rate of change and the graph of a function. In this exploration, the information about slopes of tangent lines to the graph of a function will be used to determine rules for differentiation of certain common functions.

Recall the **Definition of the Derivative:**

If $y = f(x)$ is a function, then **the derivative of y with respect to x at** $x = c$ is

Limit Defn Deriv.

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

if this limit exists. When the limit does exist, it is called the instantaneous rate of change of $y$ with respect to $x$ at $x = c$. In graphic terms, $f'(c)$ is the slope of the tangent line to the graph of $y = f(x)$ at the point $(c, f(c))$. Note: If we write $y = f(x)$, $f'(c)$ is also called the derivative of $y$ with respect to $x$ at $x = c$.

It is possible to estimate $y = f'(x)$ graphically given the graph of $y = f(x)$. Some *landmarks* of $f$ which will be helpful in sketching $f'$ follow.

1. Consider values of $x$ where the slope of the tangent is zero:
   a. Describe characteristics of the graph of a function $f$ on an interval containing $x = c$ for which the slope of the tangent through the point $(c, f(c))$ is zero. horizontal tangent
   b. If $f'(c) = 0$, what point is on the graph of $y = f'(x)$? $(c, 0)$

$f' < 0 \Leftrightarrow f$ is dec

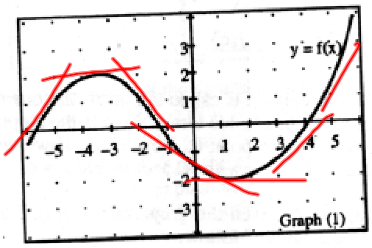
$\overbrace{\qquad}^{\text{if and only if}}$

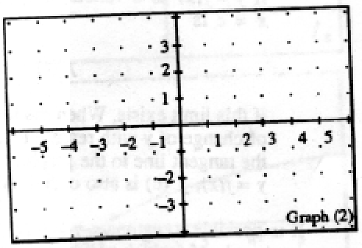
2. Consider intervals where the slopes of the tangents to $y = f(x)$ are negative:
   a. Describe the appearance of the graph of $f$ on an interval over which $f'(x) < 0$. f is decreasing
   b. If the slope of $y = f(x)$ is negative on an interval, then $f'(x)$ $\boxed{<}$ $\boxed{>}$ $\boxed{=}$ 0 (circle one). If $f'(x) < 0$ on an interval, where does the graph of $y = f'(x)$ lie with respect to the $x$-axis? BELOW
3. Consider intervals where the slopes of the tangents to $y = f(x)$ are positive.
   a. Describe the appearance of the graph of $f$ on an interval over which $f'(x) > 0$. f is increasing
   b. If the slope of $y = f(x)$ is positive on an interval, then $f'(x)$ $\boxed{<}$ $\boxed{>}$ $\boxed{=}$ 0 (circle one). Where does the graph of $y = f'(x)$ lie with respect to the $x$-axis for $x$ in such intervals?

above


## 8.2 Sketching Derivative Functions by Hand

1. Below are two pairs of axes. Use the axes (2) provided to sketch the graph of the derivative of $y = f(x)$, where $f$ is the function given in graph (1). Sketch the derivative of $y = f(x)$ as follows.
   a. Draw several tangent lines along the curve of $f$.
   b. Estimate the slope $f'$ of each tangent, using the grid marks as a guide.
   c. Plot the points $(x, f'(x))$ on graph (2).
   d. Connect the points to sketch the graph of $y = f'(x)$.



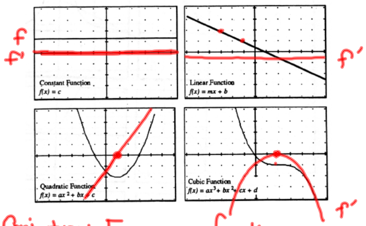



2. The graphs of several common functions $y = f(x)$ are sketched below. Complete the following for each function.

a. Sketch the graph of the derivative $f'$ on the same axes as $f$, following the suggestions given in part (1). Briefly explain why your graph is sensible.

b. Conjecture a rule to determine the derivative of the given function $y = f(x)$. Provide reasons for your choice.

c. For functions $f(x) = c$ and $f(x) = mx + b$, use the limit definition of the derivative to verify that the rule you have conjectured in part (b) is correct.



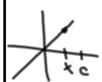

Conjecture: For power functions, the derivative appears to be one degree less.

If $f(x) = mx+b$, $f'(x) = m$

Proof:

For $f(x) = mx+b$ $\qquad$ $f(x) = K$

$$f'(x) = \lim_{c \to x} \frac{f(c) - f(x)}{c - x}$$



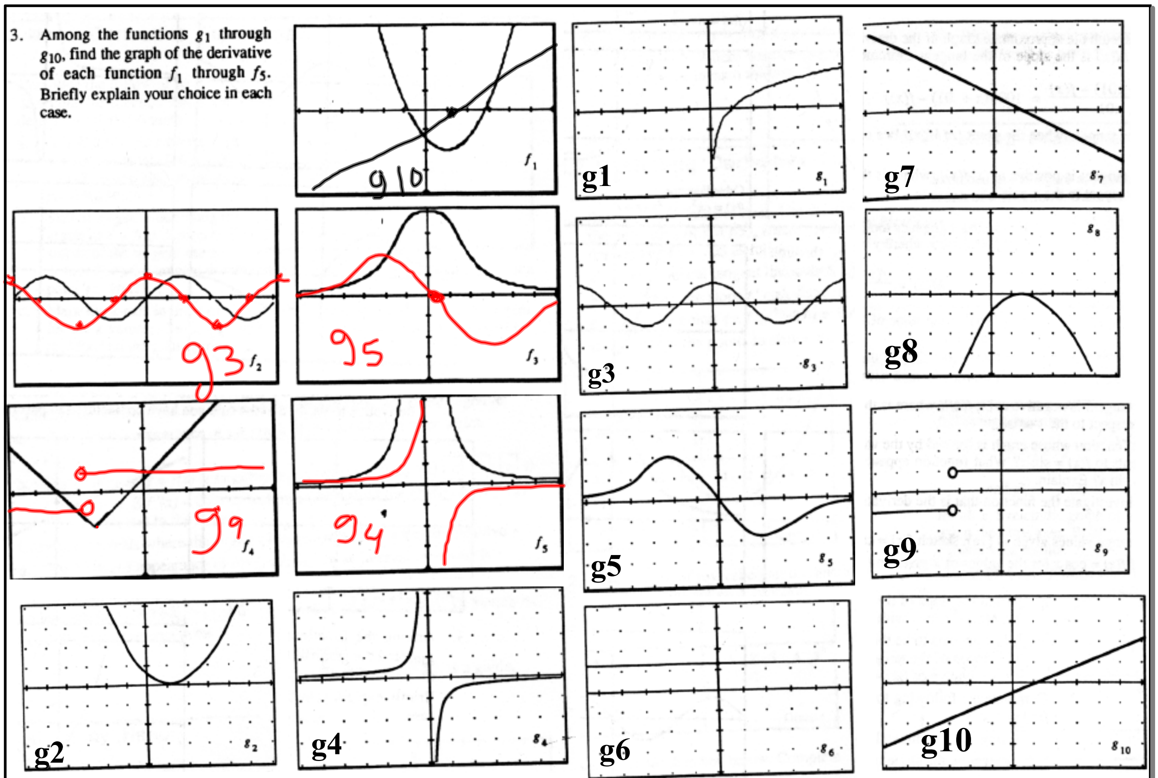

$$= \lim_{c \to x} \frac{(mc+b) - (mx+b)}{c - x}$$

$$= \lim_{c \to x} \frac{mc - mx}{c - x}$$

$$= \lim_{c \to x} \frac{m\cancel{(c-x)}}{\cancel{c-x}}$$

$$= m$$

3. Among the functions $g_1$ through $g_{10}$, find the graph of the derivative of each function $f_1$ through $f_5$. Briefly explain your choice in each case.

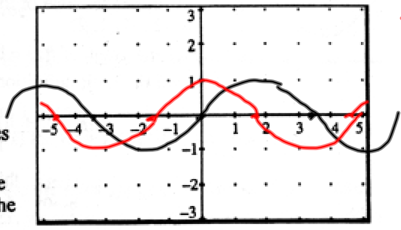


## 8.3 More Derivative Functions

Conjecture

$D_x \sin x = \cos x$

1. Complete the following to determine the function that is the derivative of $f(x) = \sin x$.



   a. Graph $f(x) = \sin x$ on the viewing rectangle $[-6.3, 6.3]$ by $[-2, 2]$ and also on the axes provided below.[1]

   b. Sketch tangent lines along the graph of $y = f(x)$. Estimate the slopes of the tangent lines drawn. For each tangent drawn along the curve of $f$ through $(x, f(x))$, plot a point $(x, M(x))$.

   c. Mark the $x$-axis in units of $\pi/4 \approx 0.785$. What is the slope of the tangent for the x values: $-3\pi/2$, $-\pi$, $-\pi/2$, 0, $\pi/2$, $\pi$, and $3\pi/2$? Mark the points $(x, M(x))$ on the graph provided.

   d. Using a graphics tool, sketch the approximate graph of the derivative of $f(x) = \sin x$ where $y = M(x)$ is the slope of the tangent estimated by

$$M(x) = \frac{f(x + .01) - f(x)}{.01} = 100(f(x + .01) - f(x)).$$

   e. What is the slope of the tangent when the point $(x, M(x))$ lies on the $x$-axis?

   f. When the slope of the tangent is positive at $(x, f(x))$, where is the point $(x, M(x))$ located with respect to the $x$-axis?

   g. When the slope of the tangent is negative at $(x, f(x))$, where is the point $(x, M(x))$ located with respect to the $x$-axis?

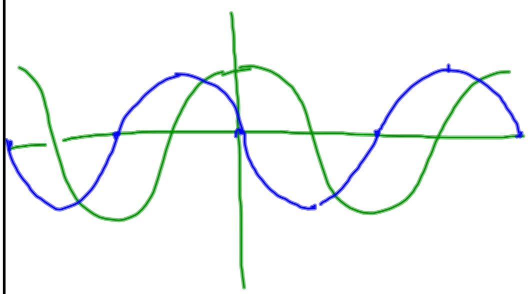
   h. What appears to be the function whose graph is formed by the values of the slopes of the tangents to $f(x) = \sin x$? What function appears to be the derivative of $f(x) = \sin x$? Explain.

2. Complete the following to investigate the function that is the derivative of $f(x) = \cos x$.

   a. Using the domain and range values given in (1a), sketch $f(x) = \cos x$.

   b. Determine the slope of $f(x) = \cos x$ for the values of $x$ given in (1c).

   c. Sketch $y = M(x)$ where $f(x) = \cos x$ (as in part (1d)). What function appears to be the derivative of $f(x) = \cos x$? Explain.



Conjecture:

$D_x \cos x = -\sin x$

3. Complete the following to investigate the function that is the derivative of $f(x) = |x|$.

a. Sketch $f(x) = |x|$ on the viewing rectangle $[-3, 3]$ by $[-3, 3]$.[1]

b. Determine the slope of the lines tangent to the graph of $f(x) = |x|$ for $x = -2, -1, 1, 2$.

c. Does the graph of $f(x) = |x|$ have a tangent line at $x = 0$? NO

d. Recall the limit definition of $f'(0)$:

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{|0+h| - 0}{h} = \lim_{h\to 0} \frac{|h|}{h}.$$

i. What is $\boxed{\lim_{h\to 0^+} \frac{|h|}{h}}$? $+1$

ii. What is $\lim_{h\to 0^-} \frac{|h|}{h}$? $= -1$

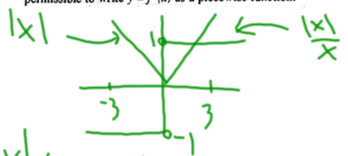
iii. What is $f'(0) = \lim_{h\to 0} \frac{|h|}{h}$?

iv. Does $f'(0)$ exist? Explain. No since side deriv limits differ

v. According to the definition of the derivative, is there a tangent line at $x = 0$? Explain.

e. Sketch $y = M(x)$ for $f(x) = |x|$. Explain the appearance of the graph of $M$ around $x = 0$. Is the graphics tool providing an accurate graph? Why or why not?

f. Write an algebraic expression for the derivative of $f(x) = |x|$. Note: It is permissible to write $y = f'(x)$ as a piecewise function.



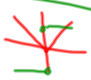

$|x|$ is continuous but not differentiable at $x=0$.

Continuity $\not\Rightarrow$ Differentiability
does not imply

But

Differentiability $\Rightarrow$ Continuity

$$D_x|x| = \frac{|x|}{x} = \frac{x}{|x|}$$



3. Consider the graphs of functions $f_1$ through $f_6$ which follow. For each function graphed

* a. sketch the graph of its derivative,

* b. sketch the graph of a function for which this function is the derivative, and

c. explain briefly how each graph was determined.

SKETCH THE FUNCTIONS WHOSE DERIVATIVES ARE SHOWN (GO BACKWARDS)