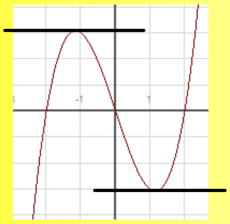
Optimization

the highs and lows of it all



Types of situations:

Minimum Cost
Maximum Profit
Least Stress
Maximum Volume
Maximum Illumination

& many, many more...

The Optimization Model

Find ___qty___ which $\genfrac{}{}{0pt}{}{\text{minimizes}}{\text{maximizes}}$ ___a second qty___ subject to ___some constraint___

you will be differentiating this quantity

That qty sometimes is in several variables.
Get it in one variable by using constraint equation.

Find______which $\genfrac{}{}{0pt}{}{\text{minimizes}}{\text{maximizes}}$ ______subject to______

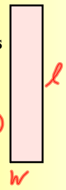
1) Find relationship between quantities (function)

2) Simplify that relationship to get a function of one variable (use constraint equation to do this).

3) $\boxed{\text{Test for extrema}}$ (absolute and/or local--depends if endpoints make sense). Use 1st or 2nd derivative test (whichever is more convenient).

Examples

A rectangle has perimeter 20 feet. What should its dimensions be if area is to be a maximum?



Find l, w which maximize $\underset{\text{area}}{A} = l \cdot w$

subject to $2l + 2w = 20$

$l + w = 10$

$A(l) = l(10-l)$

$= 10l - l^2$

$D_l A = 10 - 2l$

$10 - 2l = 0$

$l = 5$

2nd Deriv Test for L.E.

$D_l^2 A = -2 < 0 \therefore A$ is CD

$\therefore$ at $l = 5$ is local max.

and $w = 5$

$\boxed{5 \times 5}$

Suppose it requires S square inches of metal to make a cylindrical can with a top and bottom. Show that for such a can to have a maximum volume, the diameter of the base must equal the altitude.

Find r, h which maximize $V = \pi r^2 h$ subject to

$\underset{\text{constant}}{S} = 2\pi r^2 + 2\pi rh$

$\frac{S - 2\pi r^2}{2\pi r} = h$

$V = \pi r^2 \left[\frac{S - 2\pi r^2}{2\pi r}\right]$

$= \frac{1}{2}\left[Sr - 2\pi r^3\right]$

$D_r V = \frac{1}{2}S - 3\pi r^2$

$\frac{1}{2}S - 3\pi r^2 = 0$

$\frac{S}{2} = 3\pi r^2$

$S = 6\pi r^2 \therefore r^2 = \frac{S}{6\pi} \therefore r = \sqrt{\frac{S}{6\pi}}$

Second Deriv. Test Local Ext

$D_r^2 V = -6\pi r < 0$ for $r > 0$ $\therefore V$ is C[D] & local max at $r = \sqrt{\ }$

$S = 2\pi r^2 + 2\pi rh$

$S = 2\pi \cdot \frac{S}{6\pi} + 2\pi \sqrt{\frac{S}{6\pi}} \cdot h$

$r^2 = \frac{S}{6\pi} = \frac{2\pi r^2 + 2\pi rh}{6\pi}$

$r^2 = \frac{r^2}{3} + \frac{rh}{3}$

$3r^2 = r^2 + rh$

$2r^2 = rh$

$2r^2 - rh = 0$

$r(2r - h) = 0$

A trucking firm estimates that the cost/mile of a particular truck is [20+(1/5)v] cents/mile (where v is velocity in miles/hour and 20cents/hour is "fixed costs"...license, insurance, deprec, etc). The driver gets $8/hour. Suppose the truck is to be driven on a 500 mile trip. What average velocity will minimize the total cost?

A trucking firm estimates that the cost/mile of a particular truck is [20+(1/5)v] cents/mile (where v is velocity in miles/hour and 20cents/hour is "fixed costs"...license, insurance, deprec, etc). The driver gets $8/hour. Suppose the truck is to be driven on a 500 mile trip. What average velocity will minimize the total cost?

Find $\underset{\text{velocity}}{\underline{\ v\ }}$ which minimizes $\underset{\text{cost}}{\underline{\ C\ }}$ subject to all constraints.

Cost = $\frac{\text{cost}}{\text{hr}}\cdot$ hrs.

$\frac{\text{cost}}{\text{mi}}\cdot\frac{\text{mi}}{\text{hr}}+\frac{\text{driver's cost}}{\text{hr}}$

Check units!

$$C=\Big[\underbrace{\left(20+\tfrac{1}{5}v\right)\cdot v}_{\text{cents/hr}}+\underbrace{800}_{\text{cents/hr}}\Big]\cdot\underbrace{\frac{500}{v}}_{\text{hrs}}$$

$$=\left(20v+\tfrac{1}{5}v^2+800\right)\cdot\frac{500}{v}$$

$$=\left(20+\tfrac{1}{5}v+\frac{800}{v}\right)\cdot 500$$

$$D_vC=500\left(\tfrac{1}{5}+\frac{-800}{v^2}\right)$$

$$D_vC=0\Leftrightarrow\frac{1}{5}=\frac{800}{v^2},\ v^2=4000,\ v=\sqrt{4000}=20\sqrt{10}\approx 63.2\ \frac{\text{mi}}{\text{hr}}$$

$\boxed{2^{nd}\text{ Derivative Test}}$

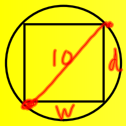
$$D_v^2C=500\left(\frac{1600}{v^3}\right)>0\ \therefore\ C\text{ is concave up}\ \therefore\ \min\text{ at }v=63.2$$

Note: 500 factored out. Length of trip doesn't affect velocity which minimized cost.

Min cost at 63.2 mph : \$226.49

at 55 mph : \$227.72

The strength of a rectangular beam is proportional to the product of its width and the square of its depth. Find the dimensions of the strongest beam that can be cut from a cylindrical log of diameter 10 inches.



Find d,w which maximizes

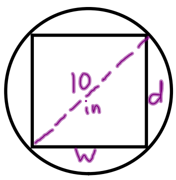
Strength = Kwd^2

subject to $d^2+w^2=100$

$$S = Kw(100-w^2)$$
$$= 100Kw - Kw^3$$
$$D_w S = 100K - 3Kw^2$$
$$=0 \text{ if } 100K = 3Kw^2$$
$$\frac{10}{\sqrt{3}} = \sqrt{\frac{100}{3}} = w$$

Test

The strength of a rectangular beam is proportional to the product of its width and the square of its depth. Find the dimensions of the strongest beam that can be cut from a cylindrical log of diameter 10 inches.



Find d,w which maximizes S strength subject to diagram.

$$S = kwd^2$$
$$w^2 + d^2 = 10^2$$
$$d^2 = 10^2 - w^2$$
$$S = k(100w - w^3)$$
$$\frac{dS}{dw} = k(100 - 3w^2)$$
$$\frac{dS}{dw} = 0 \text{ if } 100 = 3w^2 \quad \therefore w = \sqrt{\frac{100}{3}} = 5.7735 \text{ in}$$

$\boxed{\text{Second Derivative Test}}$

$$\frac{d^2S}{dw^2} = k(-6w) < 0 \quad \forall w>0 \quad \therefore S \text{ is CD}$$

$\therefore$ max at $w = 5.7735$

$$d = \sqrt{100-w^2}$$
$$= \sqrt{100 - \frac{100}{3}}$$
$$= 8.165 \text{ in}$$