Order Comparison and Ratio Tests

Jan 6-9:15 AM

Order Comparison and Ratio Tests

(ORDER) COMPARISON TEST

(for non-negative term series)

OCT

If, for all n (beyond some point)

$0 \leq a_n \leq b_n$

then, consider the series $\sum a_n$ **(subdominant)**

$\sum b_n$ **(dominant)**

If series $\sum b_n$ converges, then $\sum a_n$ must also converge

(convergence of dominant forces convergence of subdominant)

If series $\sum a_n$ diverges, then $\sum b_n$ must also diverge

(divergence of subdominant forces divergence of dominant)

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Handy things to know for comparisons:

$n! \geq 2^{n-1}$ $(n \geq 1)$

$\tan^{-1} n \ll \pi/2$

$\ln n \leq n$

$y=x$

$y=\ln x$

$\left| \begin{matrix} \sin(n) \\ \text{or} \\ \cos(n) \end{matrix} \right| \leq 1$

Try Comparison Test:

$\sum_{n=1}^{\infty} \frac{5}{3^n} = 5 \sum_{n=1}^{\infty} \frac{1}{3^n}$ converges geom, $r = \frac{1}{3} < 1$

(1) $\sum_{n=1}^{\infty} 5/(3^n + 2)$ converges. WHY?

$\frac{5}{3^n+2} \leq \frac{5}{3^n}$ because $5 \cdot 3^n \leq 5 \cdot 3^n + 10$

$\therefore$ Since $\sum_{n=1}^{\infty} \frac{5}{3^n}$ converges, so does $\sum_{n=1}^{\infty} \frac{5}{3^n+2}$ by O.C.T.

(2) $\sum_{n=1}^{\infty} (\sin^2 n)/(5^n)$

(3) $\sum_{n=126}^{\infty} \left(1/(\sqrt[3]{n} - 5) \right)$ diverges. WHY?

$\frac{1}{\sqrt[3]{n}-5} \geq \frac{1}{\sqrt[3]{n}}$

$\sum_{n=126}^{\infty} \frac{1}{\sqrt[3]{n}}$ diverges, p-series $p<1$

$\therefore$ by O.C.T.

(4) $\sum_{n=1}^{\infty} \left(n^2/(3n^4 + 5n^2 - (7/n)) \right)$

$\frac{1}{3n^2} \geq \frac{n^2}{3n^4+5n^2-\frac{7}{n}}$ since $3n^4+5n^2-\frac{7}{n} \geq 3n^4$

$\frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges p-series $p>1$

$\therefore \sum_{n=1}^{\infty} \frac{n^2}{3n^4+5n^2-\frac{7}{n}}$ conv. by O.C.T.

(5) $\sum_{n=1}^{\infty} 1/n!$

$\frac{1}{n!} \leq \frac{1}{2^{n-1}}$

$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \sum_{n=1}^{\infty} \frac{2}{2^n} = 2 \sum_{n=1}^{\infty} \frac{1}{2^n}$ conv. geom $r = \frac{1}{2} < 1$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n!}$ converges by O.C.T

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$$\sum_{n=1}^{\infty} \frac{\sin^2 n}{5^n}$$

$$\frac{\sin^2 n}{5^n} \leq \frac{1}{5^n}$$

$\sum_{n=1}^{\infty} \frac{1}{5^n}$ converges geom, $r = \frac{1}{5} < 1$

$\therefore \sum_{n=1}^{\infty} \frac{\sin^2 n}{5^n}$ converges by O.C.T.

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$$n! \ge 2^{n-1} \qquad n \ge 1$$

$$\underbrace{n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1}_{n-1 \text{ terms}} \ge \underbrace{2\cdot 2\cdot 2\cdot 2\cdots 2}_{n-1 \text{ terms}}$$

$$\frac{1}{n!} \le \frac{1}{2^{n-1}}$$

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Absolute Convergence $\Rightarrow$ **Convergence**

*(for **mixed-term** series)*

If $\sum |a_n|$ **converges, then** $\sum a_n$ **converges**

(if a series with mixed terms is converted to all positive terms by placing absolute values for each term, and that series converges, then the original mixed-term series converges)

EXAMPLE: Does $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{1}{n^2}\right)$ **converge or diverge?**

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \frac{1}{49} - \cdots$$

Compute some partial sums:

$s_1 = 1$
$s_2 = .75$
$s_3 = .8611...$
$s_4 = .798611...$
$s_5 = .838611...$
$s_6 = .810833...$
$s_7 = .83124...$
$s_8 = .81561...$
$s_9 = .8279...$

Test Absolute Convergence: $\displaystyle\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{1}{n^2} \right|$

same series as convergent p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

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RATIO TEST

Given a series $\sum a_n$

Construct the limit of the absolute value of the ratio of the n+1st term to the nth term:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

this limit will be a ~~positive~~ non-negative real number L, or ∞

(i) if $L<1$, then $\sum a_n$ converges

(ii) if $L>1$ or the limit is ∞, then $\sum a_n$ diverges

(iii) if $L=1$, then the test is inconclusive (the series could converge or diverge, ... try other tests)

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the RATIO TEST is very important; it's used in lots of situations, and it is particularly helpful for exponentials (e.g. 2^n) and factorials (e.g. n!)

$$\frac{3^{n+1}}{3^n}=\frac{3^n\cdot 3}{3^n}$$

Applications: $\lim\left|\frac{a_{n+1}}{a_n}\right|$

(1) $\sum_{n=1}^{\infty}\frac{3^n}{n!}$

Ratio Test

$$\lim_{n\to\infty}\left|\frac{3^{n+1}}{(n+1)!}\cdot\frac{n!}{3^n}\right|=\lim_{n\to\infty}\left|\frac{3}{n+1}\right|=0<1$$

$(n+1)(n)!$

$a_{n+1}\cdot\frac{1}{a_n}$

∴ series converges

(2) $\sum_{n=3}^{\infty}\left(\frac{n^2}{n^4+7}\right)$

Ratio Test

$$\lim_{n\to\infty}\left|\frac{(n+1)^2}{(n+1)^4+7}\cdot\frac{n^4+7}{n^2}\right|=1$$

inconclusive

*note: if you apply the RATIO TEST to any rational function, which is the quotient of two polynomials, you will **always** get a limit of 1 (Why?) Therefore, the RATIO TEST will be inconclusive.*

Here, you might try an ORDER COMPARISON TEST with the test series

$$\sum_{n=3}^{\infty} 1/n^2$$

(3) $\sum_{n=2}^{\infty}\frac{e^n}{n}$

Ratio Test

$$\lim_{n\to\infty}\left|\frac{e^{n+1}}{n+1}\cdot\frac{n}{e^n}\right|$$

$$=\lim_{n\to\infty}\left|e\cdot\frac{n}{n+1}\right|=e>1$$

∴ series diverges

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