(ORDER) COMPARISON TEST

(for non-negative term series)

If, for all n (beyond some point)

$$0 \leq a_n \leq b_n$$

then, consider the series $\sum a_n$ **(subdominant)**

$\sum b_n$ **(dominant)**

If series $\sum b_n$ converges, then $\sum a_n$ must also converge

(convergence of <u>dominant</u> forces convergence of <u>subdominant</u>)

If series $\sum a_n$ diverges, then $\sum b_n$ must also diverge

(divergence of <u>subdominant</u> forces divergence of <u>dominant</u>)

Handy things to know for comparisons:

$n! \geq 2^{n-1}$ $(n \geq 1)$

$\tan^{-1} n \ll \pi/2$

$\ln n \leq n$

$\left| \sin(n) \text{ or } \cos(n) \right| \leq 1$

Try Comparison Test:

(1) $\sum_{n=1}^{\infty} 5/(3^n + 2)$ **converges. WHY?**

(2) $\sum_{n=1}^{\infty} (\sin^2 n)/(5^n)$

(3) $\sum_{n=126}^{\infty} \left(1/(\sqrt[3]{n} - 5)\right)$ **diverges. WHY?**

(4) $\sum_{n=1}^{\infty} \left(n^2/(3n^4 + 5n^2 - (7/n))\right)$

(5) $\sum_{n=1}^{\infty} 1/n!$

Absolute Convergence $\Rightarrow$ Convergence
*(for **mixed-term** series)*

If $\sum |a_n|$ converges, then $\sum a_n$ converges

(if a series with mixed terms is converted to all positive terms by placing absolute values for each term, and that series converges, then the original mixed-term series converges)

EXAMPLE: Does $\sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{1}{n^2}\right)$ converge or diverge?

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \frac{1}{49} - \cdots$$

Compute some partial sums:

$s_1 = 1$
$s_2 = .75$
$s_3 = .8611...$
$s_4 = .798611...$
$s_5 = .838611...$
$s_6 = .810833...$
$s_7 = .83124...$
$s_8 = .81561...$
$s_9 = .8279...$

Test Absolute Convergence: $\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{1}{n^2} \right|$

same series as convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$

RATIO TEST

Given a series $\sum a_n$

Construct the limit of the absolute value of the ratio of the n+1^st^ term to the n^th^ term:

$$\lim \left| \frac{a_{n+1}}{a_n} \right|$$

this limit will be a positive real number L, or ∞

(i) if $L<1$, then $\sum a_n$ converges

(ii) if $L>1$ or the limit is ∞, then $\sum a_n$ diverges

(iii) if $L=1$, then the test is inconclusive
(the series could converge or diverge,
... try other tests)

the RATIO TEST is very important; it's used in lots of situations, and it is particularly helpful for exponentials (e.g. 2^n) and factorials (e.g. $n!$)

Applications:

(1) $$\sum_{n=1}^{\infty} \frac{3^n}{n!}$$

(2) $$\sum_{n=3}^{\infty} \left(\frac{n^2}{n^4+7} \right)$$

*note: if you apply the RATIO TEST to any rational function, which is the quotient of two polynomials, you will **always** get a limit of 1 (Why?) Therefore, the RATIO TEST will be inconclusive.*

Here, you might try an ORDER COMPARISON TEST with the test series

$$\sum_{n=3}^{\infty} 1/n^2$$

(3) $$\sum_{n=2}^{\infty} \frac{e^n}{n}$$