

Prelude to Today's Technique $\int \frac{3}{x+4} dx = 3 \int \frac{1}{x+4} dx$

12) Add the two fractions (and get a common denominator)

$$\frac{3}{x+4} + \frac{2}{x-1} = \frac{5x+5}{x^2+3x-4}$$

13) For the above problem, which side would be easier to integrate? **LEFT**

14) Try to break down $\frac{6x+13}{x^2+x-6}$ into the sum of two functions

(Hint: factor the denominator first)

Need help? Read on.

$$\int \frac{6x+13}{x^2+x-6} dx = \int \frac{1}{x+3} + \frac{5}{x-2} dx = \ln|x+3| + 5\ln|x-2| + C$$

15) For each linear factor in the denominator, there is an addend in the sum of the form $\frac{\text{constant}}{\text{linear factor}}$

So... for the above problem, $\frac{6x+13}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$ Now you solve for A and B!

$$6x+13 = A(x-2) + B(x+3)$$

$$= Ax - 2A + Bx + 3B$$

$$= (A+B)x - 2A + 3B$$

$A+B=6$
 $-2A+3B=13$
 $A=1$
 $B=5$

16) For each repeated linear factor in the denominator, the decomposition contains a sum of partial fractions (one for each power)

So... for $\frac{3x^2-7x+10}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$

Note the right hand side set up. Solve for

A= _____
 B= _____
 C= _____

Integration Technique #5

Partial Fraction Decomposition (PFD)

-For integrating rational functions (what are those?)

Any rational function can be integrated by decomposing it into a sum of simpler (partial) fractions...each of which can be integrated by earlier techniques. The results appear as polynomials, rational functions arctan and ln.

$$\int \frac{P(x)}{Q(x)} dx \Rightarrow \int \frac{A}{(x+a)^k} + \frac{(Bx+C)}{(x^2+bx+c)^m} dx$$

This technique has its roots in two theorems of Algebra:

1) Every polynomial with real coefficients can be expressed as a product of linear and/or quadratic factors with real coefficients

2) Every proper rational expression may be expressed as a finite sum of partial fractions of the form:

$$\frac{A}{(x+a)^k}$$

and

$$\frac{(Bx+C)}{(x^2+bx+c)^m}$$

irreducible quadratic

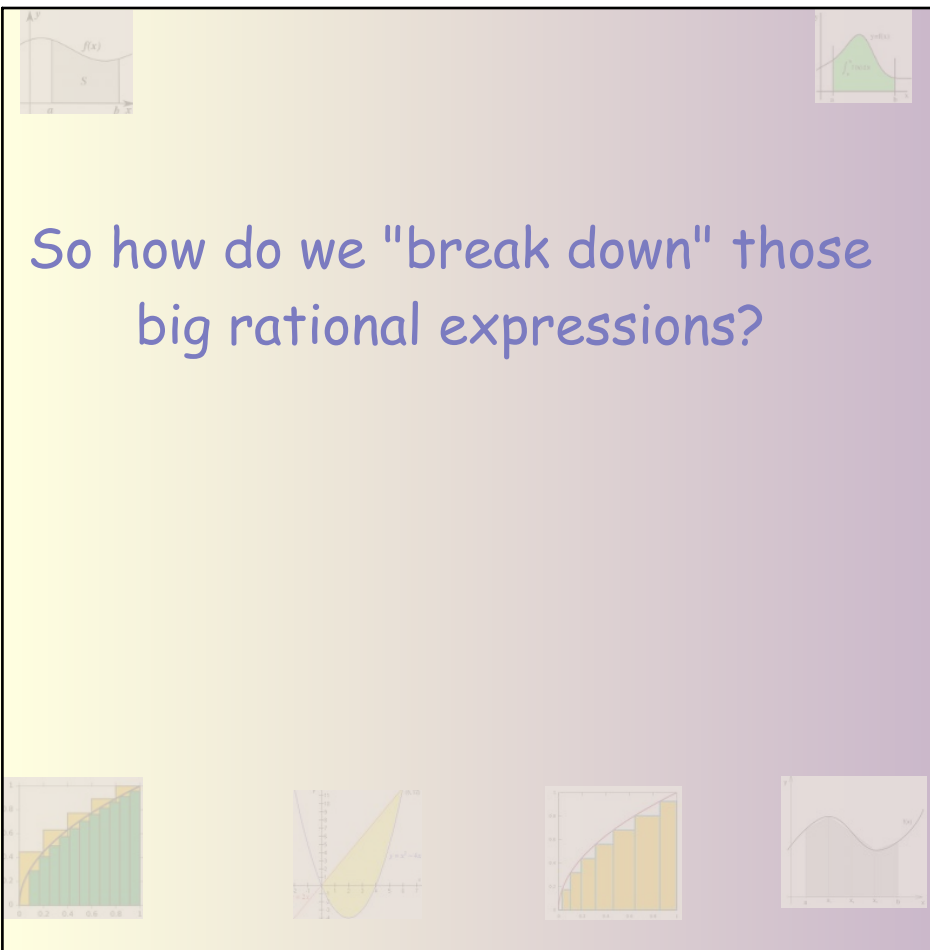
Consider the partial fractions

$$\frac{2}{x-1} + \frac{3}{x+3} = \frac{5x+3}{(x-1)(x+3)}$$

$$\int \frac{5x+3}{(x-1)(x+3)} dx = \int \frac{2}{x-1} + \frac{3}{x+3} dx$$

which integral is manageable?

So how do we "break down" those big rational expressions?



Here's the DECOMPOSITION algorithm:

1) Always first reduce the rational expression to proper form (degree num < degree den). To do so, you may have to long divide.

2) For each factor in the denominator of the form $(px+q)^m$ where $m \geq 1$ the decomposition contains a sum of m partial fractions of the form (one for each power):

$$\frac{A_1}{px+q} + \frac{A_2}{(px+q)^2} + \frac{A_3}{(px+q)^3} + \dots + \frac{A_m}{(px+q)^m}$$

~~3) For each factor in the denominator of the form $(ax^2+bx+c)^n$ where $n \geq 1$, (irreducible quadratic... $b^2-4ac < 0$), the decomposition contains a sum of n partial fractions of the form:~~

~~$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \dots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$$~~

Example

$$\int \frac{x^3+3x}{x^2-2x-3} dx \quad \text{improper so divide first...} \quad \int x+2 + \frac{10x+6}{x^2-2x-3} dx$$

Then decompose

$$\frac{10x+6}{x^2-2x-3} = \frac{A}{x-3} + \frac{B}{x+1}$$

$$10x+6 = A(x+1) + B(x-3)$$

$$= (A+B)x + A - 3B$$

$$A+B=10$$

$$A-3B=6$$

$$B=1$$

$$A=9$$

Then integrate

$$\int \frac{x^3+3x}{x^2-2x-3} dx$$

$$\int (x+2) + \frac{9}{x-3} + \frac{1}{x+1} dx$$

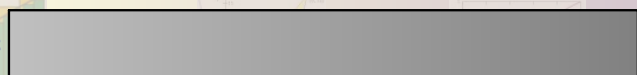
$$\frac{1}{2}x^2 + 2x + 9\ln|x-3| + \ln|x+1| + C$$

Nonessential example

$$\int \frac{x^4 - x^3 + 2x^2 - x + 2}{(x-1)(x^2+2)^2} dx$$



Answer:



Example

$$\int \frac{x^2 + 2x + 3}{(x-1)(x+1)^2} dx$$

$$\frac{x^2 + 2x + 3}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$\begin{aligned} x^2 + 2x + 3 &= A(x+1)^2 + B(x-1)(x+1) + C(x-1) \\ &= A(x^2 + 2x + 1) + B(x^2 - 1) + C(x-1) \\ &= Ax^2 + Bx^2 + 2Ax + Cx + A - B - C \end{aligned}$$

$$\begin{aligned} A + B &= 1 \\ 2A + C &= 2 \\ A - B - C &= 3 \end{aligned}$$

SIMULT on TI-89 is nice :)

$$\begin{aligned} A &= 1 - B \\ 2(1 - B) + C &= 2 \\ 1 - B - B - C &= 3 \end{aligned}$$

$$2 - 2B + C = 2$$

$$-2B + C = 0$$

$$-2B - C = 2$$

$$-4B = 2$$

$$B = -\frac{1}{2}$$

$$A = \frac{3}{2}$$

$$C = -1$$

$$\frac{\frac{3}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1} + \frac{-1}{(x+1)^2}$$

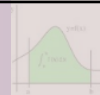
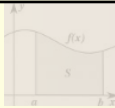
$$\frac{-3}{2} \ln|x-1| - \frac{1}{2} \ln|x+1| + \frac{1}{x+1} + C$$

$$\int \frac{-1}{(x+1)^2} dx = \int \frac{-1}{w^2} dw$$

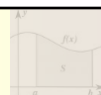
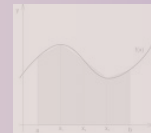
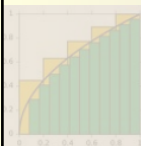
$$w = x+1$$

$$dw = dx$$

$$\int -w^{-2} dw = -\left(-\frac{1}{w}\right)$$



Homework for Friday:
10.4 # 1,7,17,23 and
worksheet front (we
did the back in class as
a prelude)



$$\int \cot^4 x dx = \int (\csc^2 x - 1)^2 dx$$



$$= \int \csc^4 x - 2\csc^2 x + 1 dx$$

$$= \int \csc^4 x dx - 2 \int \csc^2 x dx + \int dx$$

$$= \int \csc^2 x \csc^2 x dx \dots$$

$$= \int \csc^2 x (\cot^2 x + 1) dx$$

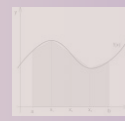
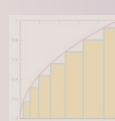
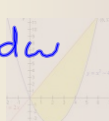
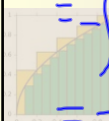
$$w = \cot x$$

$$-dw = \csc^2 x dx$$

$$= - \int w^2 + 1 dw$$

$$= -\frac{1}{3}w^3 - w$$

$$= -\frac{1}{3}\cot^3 x - \cot x + 2\cot x + x + C$$



Homework Answers (Integration Technique w/ksht)

$$1) \frac{-\sqrt{x^2+49}}{49x} + C$$

trig sub

$$2) \frac{5\sqrt{2}}{12}$$

trig id.

$$3) 2\sin\sqrt{x} + C$$

simple sub

$$4) \frac{1}{5}\sec^5 x - \frac{2}{3}\sec^3 x + \sec x + C$$

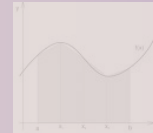
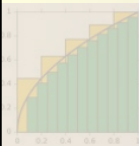
trig id.

$$5) \frac{x \cdot 2^x - 2^x}{\ln 2 \ln^2 2} + C$$

parts

$$6) \int_0^2 x e^{x^2} dx = \frac{1}{2}e^4 - \frac{1}{2}$$

simple sub



$$7) \int_0^3 \frac{x}{3x^2+4} dx$$

simple sub

$$\frac{1}{18} \ln \frac{31}{4}$$

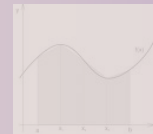
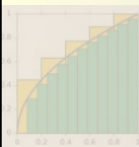
$$8) \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}[x - \tan^{-1} x] + C$$


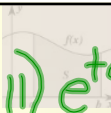
parts, trig sub, trig id.

$$9) \frac{5\pi}{16}$$

$$10) \frac{3}{7}(w+4)^{7/3} - 3(w+4)^{4/3} + C$$

simple sub



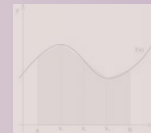
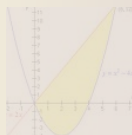
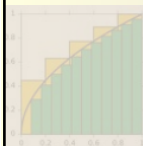


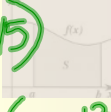
11) $e^{\tan x} + C$
simple sub

12) $\frac{5x+5}{(x+4)(x-1)}$


13) Left side !!

14) *in*



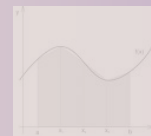
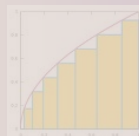
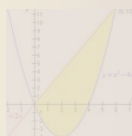
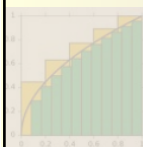


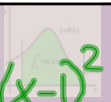

15) $\frac{6x+13}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$


$$\begin{aligned} 6x+13 &= A(x-2) + B(x+3) \\ 6x+13 &= (A+B)x - 2A+3B \\ A+B &= 6 \\ -2A+3B &= 13 \end{aligned} \quad \therefore \begin{aligned} A &= 1 \\ B &= 5 \end{aligned}$$

$$\int \frac{6x+13}{(x+3)(x-2)} dx = \int \quad dx$$

=




$$16) \quad 3x^2 - 7x + 10 = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

$$= A(x^2 + x - 2) + Bx + 2B + C(x^2 - 2x + 1)$$

$$= Ax^2 + Cx^2 + Ax + Bx - 2Cx - 2A + 2B + C$$

$$\therefore A + C = 3$$

$$A + B - 2C = -7$$

$$-2A + 2B + C = 10$$

$$A = -1$$

$$B = 2$$

$$C = 4$$

