Prelude to Today's Technique

$$\int \frac{3}{x+4}\,dx = 3\int \frac{1}{x+4}\,dx$$

$w = x+4$

$dw = 1\,dx$

$$3\int \frac{1}{w}\,dw = 3\ln|x+4|+C$$

12) Add the two fractions (and get a common denominator)

$$\frac{3}{x+4} + \frac{2}{x-1} = \frac{5x+5}{x^2+3x-4}$$

$$\frac{3(x-1)+2(x+4)}{(x+4)(x-1)}$$

13) For the above problem, which side would be easier to integrate? LEFT

14) Try to break down $\frac{6x+13}{x^2+x-6}$ into the sum of two functions

$(x+3)(x-2)$

(Hint: factor the denominator first)

Need help? Read on.

$$\int \frac{6x+13}{x^2+x-6}\,dx = \int \frac{1}{x+3} + \frac{5}{x-2}\,dx = \ln|x+3| + 5\ln|x-2| + C$$

15) For each linear factor in the denominator, there is an addend in the sum of the form $\frac{\text{constant}}{\text{that linear factor}}$

So... for the above problem, $\frac{6x+13}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$

Now you solve for A and B!

$6x+13 = A(x-2) + B(x+3)$

$= Ax - 2A + Bx + 3B$

$= (A+B)x - 2A + 3B$

$A+B=6$

$-2A+3B=13$

$A=1 \quad B=5$

A= 1

B= 5

16) For each repeated linear factor in the denominator, the decomposition contains a sum of partial fractions (one for each power)

So.... for $\frac{3x^2-7x+10}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}$

Note the right hand side set up. Solve for

A= ________

B= ________

C= ________

Integration Technique #5

Partial Fraction Decomposition (PFD)

-For integrating rational functions (what are those?)

Any rational function can be integrated by decomposing it into a sum of simpler (partial) fractions...each of which can be integrated by earlier techniques. The results appear as polynomials, rational functions arctan and ln.

$$\int \frac{P(x)}{Q(x)}\,dx \Rightarrow \int \frac{A}{(x+a)^k} + \frac{(Bx+C)}{(x^2+bx+c)^m}\,dx$$

This technique has its roots in two theorems of Algebra:

1) Every polynomial with real coefficients can be expressed as a product of linear and/or quadratic factors with real coefficients

2) Every proper rational expression may be expressed as a finite sum of partial fractions of the form:

$$\frac{A}{(x+a)^k} \quad \text{and} \quad \frac{(Bx+C)}{(x^2+bx+c)^m}$$

irreducible quadratic

Consider the partial fractions

$$\frac{2}{x-1}+\frac{3}{x+3}=\frac{5x+3}{(x-1)(x+3)}$$

$$\int \frac{5x+3}{(x-1)(x+3)}dx = \int \frac{2}{x-1}+\frac{3}{x+3}\,dx$$

which integral is manageable?

So how do we "break down" those big rational expressions?

Here's the DECOMPOSITION algorithm:

1) Always first reduce the rational expression to proper form (degree num<degree den). To do so , you may have to long divide.

2) For each factor in the denominator of the form $(px+q)^m$ where $m\geq 1$ the decomposition contains a sum of m partial fractions of the form (one for each power):

$$\frac{A_1}{px+q}+\frac{A_2}{(px+q)^2}+\frac{A_3}{(px+q)^3}+\ldots\frac{A_m}{(px+q)^m}$$

~~3) For each factor in the denominator of the form $(ax^2+bx+c)^n$ where $n\geq 1$, (irreducible quadratic...$b^2-4ac<0$), the decomposition contains a sum of n partial fractions of the form:~~

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\ldots\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

Example

$$\int\frac{x^3+3x}{x^2-2x-3}dx$$ improper so divide first... $$\int x+2+\frac{10x+6}{x^2-2x-3}dx$$

Then decompose

$$\frac{10x+6}{x^2-2x-3}=\frac{A}{(x-3)}+\frac{B}{(x+1)}$$

$$10x+6=A(x+1)+B(x-3)$$
$$=(A+B)x+A-3B$$

$$A+B=10$$
$$A-3B=6$$

$$B=1$$
$$A=9$$

Then integrate

$$\int\frac{x^3+3x}{x^2-2x-3}dx$$

$$\int(x+2)+\frac{9}{x-3}+\frac{1}{x+1}dx$$

$$\frac{1}{2}x^2+2x+9\ln|x-3|+\ln|x+1|+C$$

Nonessential example

$$\int \frac{x^4 - x^3 + 2x^2 - x + 2}{(x-1)(x^2+2)^2}dx$$

Answer:

Example

$$\int \frac{x^2 + 2x + 3}{(x-1)(x+1)^2}dx$$

$$\frac{x^2+2x+3}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$\begin{aligned} x^2+2x+3 &= A(x+1)^2 + B(x-1)(x+1) + C(x-1) \\ &= A(x^2+2x+1) + B(x^2-1) + C(x-1) \\ &= Ax^2 + Bx^2 + 2Ax + Cx + A - B - C \end{aligned}$$

$A+B=1$

$2A+C=2$

$A-B-C=3$

SIMULT on TI-89 is nice ☺

$A=1-B$

$2(1-B)+C=2$

$1-B-B-C=3$

$2-2B+C=2$

$-2B+C=0$

$-2B-C=2$

$-4B=2$

$B=-\frac{1}{2}$

$A=\frac{3}{2}$

$C=-1$

$$\overset{\frac{3}{2}}{\frac{A}{x-1}} + \overset{-\frac{1}{2}}{\frac{B}{x+1}} + \overset{-1}{\frac{C}{(x+1)^2}}$$

$$-\frac{3}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + \frac{1}{x+1} + C$$

$$\int \frac{-1}{(x+1)^2}dx = \int -\frac{1}{w^2}dw$$

$w = x+1$

$dw = dx$

$$\int -w^{-2}dw = -\left(-\frac{1}{w}\right)$$

Homework for Friday:
10.4 # 1,7,17,23 and
worksheet front (we
did the back in class as
a prelude)

$$\int \cot^4 x\,dx = \int (\csc^2 x - 1)^2\,dx$$

$$= \int \csc^4 x - 2\csc^2 x + 1\,dx$$

$$= \int \csc^4 x\,dx - 2\int \csc^2 x\,dx + \int 1\,dx$$

$$= \int \csc^2 x \csc^2 x\,dx \quad \ldots$$

$$= \int \csc^2 x(\cot^2 x + 1)\,dx$$

$$w = \cot x$$

$$-dw = \csc^2 x\,dx$$

$$= -\int w^2 + 1\,dw$$

$$= -\frac{1}{3}w^3 - w$$

$$= -\frac{1}{3}\cot^3 x - \cot x + 2\cot x + x + C$$

Homework Answers (Integration Technique Wksht)

1) $-\frac{\sqrt{x^2+49}}{49x}+C$
trig sub

2) $\frac{5\sqrt{2}}{12}$
trig id.

3) $2\sin\sqrt{x}+C$
simple sub

4) $\frac{1}{5}\sec^5 x-\frac{2}{3}\sec^3 x+\sec x+C$
trig id.

5) $\frac{x\cdot 2^x}{\ln 2}-\frac{2^x}{\ln^2 2}+C$
parts

6) $\int_0^2 xe^{x^2}dx=\frac{1}{2}e^4-\frac{1}{2}$
simple sub

7) $\frac{\int_0^3 \frac{x}{3x^2+4}}{3}$ simple sub $\frac{1}{18}\ln\frac{31}{4}$

8) $\frac{1}{2}x^2\tan^{-1}x-\frac{1}{2}\left[x-\tan^{-1}x\right]+C$
parts, trig sub, trig id.

9) $\frac{5\pi}{16}$

10) $\frac{3}{7}(w+4)^{7/3}-3(w+4)^{4/3}+C$
Simple sub

11) $e^{\tan x} + C$

simple sub

12) $\dfrac{5x+5}{(x+4)(x-1)}$

13) Left side!!

14)

15)

$$\frac{6x+13}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$$

$$6x+13 = A(x-2) + B(x+3)$$

$$6x+13 = (A+B)x - 2A + 3B$$

$$\begin{aligned} A+B &= 6 \\ -2A+3B &= 13 \end{aligned} \qquad \therefore \begin{aligned} A &= 1 \\ B &= 5 \end{aligned}$$

$$\int \frac{6x+13}{(x+3)(x-2)}\,dx = \int \qquad dx$$

$$=$$

16) $3x^2 - 7x + 10 = A(x-1)(x+2) + B(x+2) + C(x-1)^2$

$= A(x^2+x-2) + Bx + 2B + C(x^2 - 2x + 1)$

$= Ax^2 + Cx^2 + Ax + Bx - 2Cx - 2A + 2B + C$

$\therefore \quad A + C = 3$

$A + B - 2C = -7$

$-2A + 2B + C = 10$

$A = -1$

$B = 2$

$C = 4$