"Now here is a difficulty that really is great.
It may happen that a finite equation may be expressed as an infinite one, so that the equation obtained may really be the same as the given equation although it does not appear to be.

For example,

$$\frac{x}{1+x} = x - x^2 + x^3 - x^4 + x^5 - x^6 + \ldots"$$

Leibniz (1646-1716)

Power Series

POWER SERIES

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots$$

(center at 0)

$$\sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 \ldots + a_n(x-a)^n \ldots$$

(center at a)

A *power series* is a <u>function</u> (like an infinite polynomial).

Its <u>domain</u> is the set of all x for which the series <u>converges.</u>

When you replace "x" by a real number, (assuming the center "a" is known, and the "a_n" is an expression in "n"), you get a "constant-term" series, and you can use the convergence tests you have already learned to see if the series converges for that particular x.

Its <u>range</u> is the set of all sums (for varying x's in the domain) for which this series converges.

RADIUS and INTERVAL of CONVERGENCE

Consider the power series $\sum_{n=1}^{\infty} (x - 1)^n$

(this series is centered at 1)

Problem: Find all x for which this series converges.

Apply the **RATIO TEST**

$$\lim_{n \to \infty} \left| \frac{n+1^{st} \text{ term}}{n^{th} \text{ term}} \right| = \lim_{n \to \infty} \left| \frac{(x-1)^{n+1}}{(x-1)^n} \right| = \lim_{n \to \infty} |x - 1| = x - 1$$

How do we interpret this?

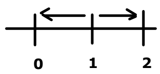
Remember we have convergence using the ratio test if the limit comes out < 1

So the series $\sum_{n=1}^{\infty} (x - 1)^n$ converges for those x such that

$$|x - 1| < 1$$

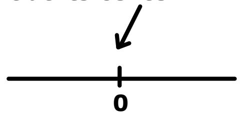
that is, all x such that $-1 < x-1 < 1$ or $x \in (0, 2)$

(0 , 2) is the basic interval of convergence, and since the series is centered at 1, the radius of convergence is r = 1

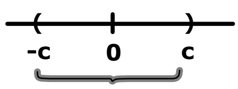


NATURE OF POWER SERIES CONVERGENCE/DIVERGENCE

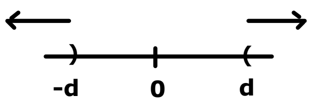
A power series $\sum_{n=1}^{\infty} x^n$ <u>always</u> converges right at its center



If a power series <u>converges</u> at any point c, then it converges for all x in a symmetric open interval about its center



If a power series <u>diverges</u> at any point d, then it diverges for all x that are d or more units <u>away</u> from its center



The largest positive number r (established by the RATIO TEST) such that the series converges for all x where

$$|x - \text{center}| < r$$

is called the ***radius of convergence*** of the power series

Use the RATIO TEST to find the *radius of convergence* (Note: if you test endpoints individually, you might be able to determine the exact *interval of convergence.)*

$$\sum_{n=0}^{\infty} \left(\frac{1}{(n+1)}\right) x^n$$

$$\sum_{n=0}^{\infty} (-1)^{n+1} \frac{n+1}{n!} x^n$$

A Famous Series: $$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sum_{n=1}^{\infty} \frac{(5x)^{2n}}{n\,(4^{2n})}$$

$$\sum_{n=0}^{\infty} n!\, x^n$$

Power Series Convergence (Numerically)

Recall $$\sum_{n=1}^{\infty} (x-1)^n = (x-1) + (x-1)^2 + (x-1)^3 + \ldots$$

This power series has center 1 and radius of convergence 1, so it converges for all x in the interval (0,2). Have a look at several points and compute the first 7 successive partial sums:

suppose x = 0.7	suppose x = 1.2	suppose x = 2.7
S_1 = -.3	S_1 = .2	S_1 = 1.7
S_2 = -.21	S_2 = .24	S_2 = 4.59
S_3 = -.237	S_3 = .248	S_3 = 9.503
S_4 = -.2289	S_4 = .2496	S_4 = 17.8551
S_5 = -.23133	S_5 = .24992	S_5 = 32.05367
S_6 = -.230601	S_6 = .249984	S_6 = 56.191239
S_7 = -.2308197	S_7 = .2499968	S_7 = 97.2251063

In each case, do you suspect convergence or divergence?

SKIP FOR NOW

Power Series Convergence (Graphically)

In your calculator graphing menu, define the functions:

$y_1 = (x - 1)$

$y_2 = y_1 + (y_1)^2$

$y_3 = y_2 + (y_1)^3$

$y_4 = y_3 + (y_1)^4$

$y_5 = y_4 + (y_1)^5$

$y_6 = y_5 + (y_1)^6$

$y_7 = y_6 + (y_1)^7$

Graph these curves in a window $[-1,3]_x$ by $[-4,4]_y$
What do you conclude?

Now graph the function $f(x) = \dfrac{(x - 1)}{(2 - x)}$

An Example of a function represented by a series

Consider the series $\displaystyle\sum_{n=1}^{\infty} (x - 1)^n$

THINK of this as a <u>geometric</u> series, with first term $(x - 1)$, and common ratio $(x - 1)$

The geometric series converges if and only if
$|x - 1| < 1$ so it converges for all $x \in (0 , 2)$

(Notice: this is the <u>same</u> conclusion as the RATIO TEST)

What does it converge to? $\dfrac{\text{first term}}{1 - \text{common ratio}} = \dfrac{x - 1}{1 - (x - 1)}$

which is precisely the function $f(x) = \dfrac{x - 1}{2 - x}$

*The function and the series have the **<u>same</u>** value (function value for the function, sum of series for the series) for all x in the open interval from 0 to 2. This is what it means for a power series to represent a function over its interval of convergence.*

WE SAY THE <u>SERIES</u> $\sum_{n=1}^{\infty} (x - 1)^n$ **REPRESENTS**

THE <u>FUNCTION</u> $f(x) = \dfrac{x - 1}{2 - x}$

OVER THE INTERVAL (0 , 2)

This means that if you select any real number x in the open interval between 0 and 2, and replace that for "x" in the series...you'll get the same value (sum to which the series converges)as if you were to replace the "x" in the function.

check: let x =1.5

<u>series</u> becomes $\sum_{n=1}^{\infty} (.5)^n$ *which converges to 1*

<u>function</u> $f(1.5) = \dfrac{1.5 - 1}{2 - 1.5} = 1$

Back to the Leibniz Quote

Leibniz observed: $\dfrac{x}{1 + x} = x - x^2 + x^3 - x^4 + x^5 - x^6 + \ldots$

<u>Reasoning like geometric series</u>: (look at the right side of the equation) if "x" is the first term, and the common ratio is "-x", then the series converges to $\dfrac{x}{1 - (-x)}$ **which is** $\dfrac{x}{1 + x}$

for all x where the series actually converges ... $|-x| < 1$, **or all** $x \in (-1, 1)$

<u>Reasoning by making certain adjustments:</u>

(1) Take the series for $\dfrac{1}{1 - x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots$

(2) Replace "x" by "-x" $\dfrac{1}{1 + x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots$

(3) Finally, multiply the function and the series by "x"

$$\frac{x}{1 + x} = x - x^2 + x^3 - x^4 + x^5 + \ldots$$

Can you do these adjustments without changing the radius of convergence? We'll see later that we <u>can!</u> But *some* "adjustments" have an effect on the radius of convergence.

We just saw that the function

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots$$

(for all $x \in (-1, 1)$)

Now, replace "x" by "$-x^2$"

we obtain $$\frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \ldots$$

Now, integrate the function (and the series)

$$\int \frac{1}{1 + x^2}\,dx = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots$$

Have you seen this integral (and this series) before?

Some Really Helpful Series to Know...

next slide

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{\_\_\_}\_\_\_ = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{\_\_\_}\_\_\_ = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$e^x =$$

$$\frac{1}{1-x} =$$

Here's something interesting... getting the series for $\ln x$

We know $\int \frac{1}{x} dx = \ln x$

$$\frac{1}{1-(1-x)} = \frac{1}{x} \quad \text{(express } \tfrac{1}{x} \text{ "geometrically")}$$

$$\sum_{n=0}^{\infty} 1(1-x)^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \dots$$

So... $\int \frac{1}{x} dx = \int \left(1 - (x-1) + (x-1)^2 - (x-1)^3 + \dots\right) dx$

So $\ln x = x - \frac{(x-1)}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$