Recall

$$\frac{x^2-a^2}{x-a} = x+a$$

$$\begin{array}{r} x+a \\ x-a \overline{\big)\, x^2 \qquad -a^2} \\ -\,x^2+ax \\ \hline ax-a^2 \\ -\,ax+a^2 \\ \hline \end{array}$$

$$\frac{x^3-a^3}{x-a} = x^2+ax+a^2$$

$$\begin{array}{r} x^2+ax+a^2 \\ x-a \overline{\big)\, x^3 \qquad\qquad -a^3} \\ -\,x^3+ax^2 \\ \hline ax^2 \\ -\,ax^2+a^2x \\ \hline a^2x-a^3 \end{array}$$

$$\frac{x^n-a^n}{x-a} = x^{n-1}+ax^{n-2}+a^2x^{n-3}+\dots+a^{n-1}$$

$$\begin{array}{r} x^{n-1}+ax^{n-2}+a^2x^{n-3}+a^3x^{n-4}+\dots a^{n-1} \\ x-a \overline{\big)\, x^n \qquad\qquad\qquad\qquad -a^n} \\ -\,x^n+ax^{n-1} \\ \hline ax^{n-1} \\ -\,ax^{n-1}+a^2x^{n-2} \\ \hline a^2x^{n-2} \\ -\,a^2x^{n-2}+a^3x^{n-3} \\ \hline a^3x^{n-3} \end{array}$$

Sooooooo

We have conjectured many lovely things in calculus recently. . .

But the loveliest of these things

Power Rule

We know

$f(x) = x^2, \quad f'(x) = 2x$

$g(x) = x^3, \quad g'(x) = 3x^2$

Conjecture

If $h(x) = ax^n$

Then $h'(x) = a \cdot nx^{n-1}$

Proof of Power Rule

$f(x) = x^n$

limit definition of derivative.

$$f'(x) = \lim_{c \to x} \frac{f(x) - f(c)}{x - c}$$

$$= \lim_{c \to x} \frac{x^n - c^n}{x - c}$$

$$= \lim_{c \to x} \left[\underbrace{x^{n-1} + c\,x^{n-2} + c^2 x^{n-3} + \ldots + c^{n-1}}_{n \text{ terms}} \right]$$

$$= x^{n-1} + x \cdot x^{n-2} + x^2 \cdot x^{n-3} + \ldots + x^{n-1}$$

$$= x^{n-1} + x^{n-1} + x^{n-1} + \ldots x^{n-1}$$

$$= n x^{n-1}$$

THE POWER RULE!

If $f(x) = x^n$ ← power function

then

$$f'(x) = nx^{n-1}$$

$n \neq 0$

what is your conjecture based on your recent work (or reading)?

Let $f(x)=x^n$. What is the derivative function? (algebraically)

Power Function

$$\lim_{a\to x}\frac{f(x)-f(a)}{x-a}$$

$$\lim_{a\to x}\frac{x^n-a^n}{x-a}$$

$$\lim_{a\to x}\left[x^{n-1}+ax^{n-2}+a^2x^{n-3}+a^3x^{n-4}+\dots+a^{n-2}x+a^{n-1}\right]$$

n terms

$$\underbrace{x^{n-1}}+\underbrace{x\cdot x^{n-2}}+\underbrace{x^2\cdot x^{n-3}}+\underbrace{x^3\cdot x^{n-4}}+\dots+\underbrace{x^{n-1}}$$

$$= x^{n-1}+x^{n-1}+x^{n-1}+x^{n-1}+\dots+x^{n-1}$$

$$= nx^{n-1}$$

$$\boxed{\text{If } f(x)=x^n \text{ then } f'(x)=nx^{n-1}}$$

Power Rule.

Examples

1) Let $m(n)=9n^5$

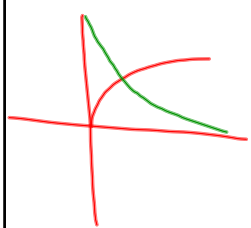
$m'(n)=\underline{45n^4}$

2) Let $p(q)=\sqrt{q}=q^{\frac{1}{2}}$

$p'(q)=\underline{\frac{1}{2}q^{-\frac{1}{2}}}=\frac{1}{2q^{\frac{1}{2}}}=\frac{1}{2\sqrt{q}}$

3) Let $s(t)=5t^4-2t+7$

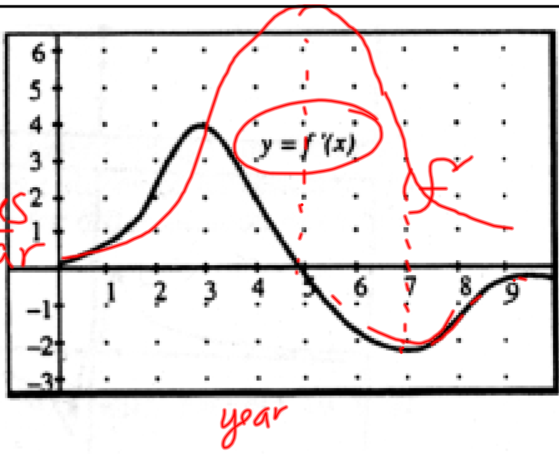
$s'(t)=\underline{20t^3-2+0}$



Because the deriv. is a limit (of stuff) you can take the deriv of a sum + get the sum of derivatives.

2. The graph of $y = f'(x)$ is shown in the figure at right. Assume that the function $y = f(x)$ represents the number of sales of an innovative calculus workbook in its first several years. Note that this is **the graph of the derivative of f, not the graph of the function f itself.**

\# sales / year



a. What does f' represent in this situation?

b. Estimate for which year(s) sales were at a maximum. Explain. → 5

c. Estimate for which year(s) the least number of workbooks were sold? Explain. 10 years or initially

3 d. During which year(s) were the sales increasing most rapidly? Explain.

e. If you were the publisher, would you have chosen to carry the workbook for the number of years represented in the graph? Explain.

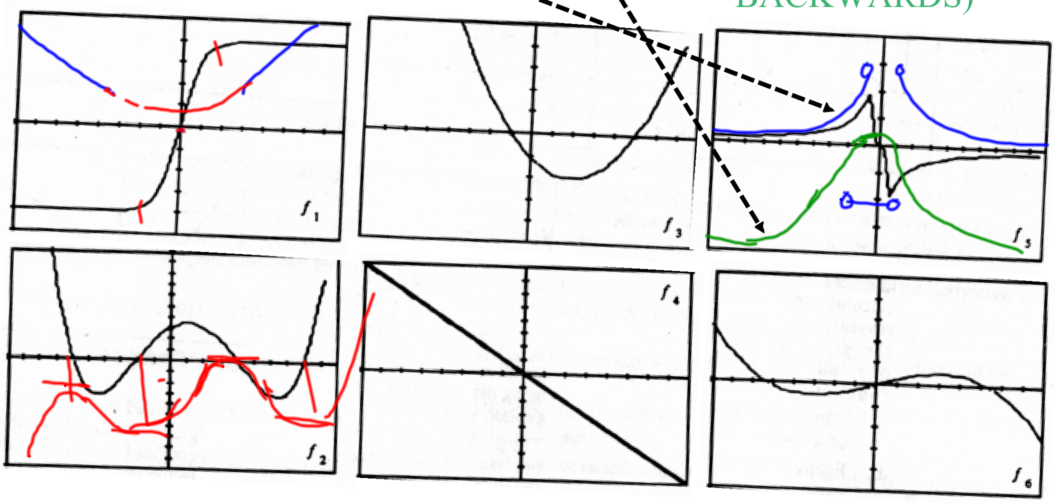
3. Consider the graphs of functions f_1 through f_6 which follow. For each function graphed

a. sketch the graph of its derivative,

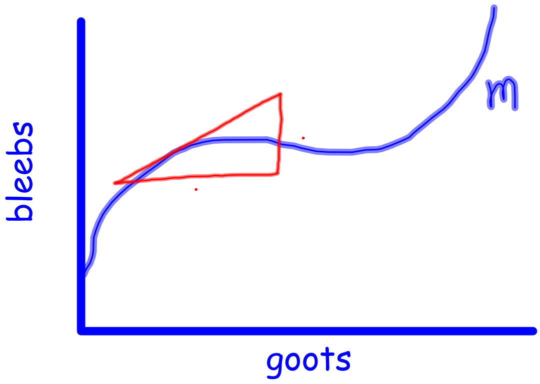
b. sketch the graph of a function for which this function is the derivative, and

c. explain briefly how each graph was determined.

SKETCH THE FUNCTIONS WHOSE DERIVATIVES ARE SHOWN (GO BACKWARDS)



Applications (2.5)



What are the units of m'?

$\frac{\text{bleeb}}{\text{goot}}$

The cost (\$) to produce w cars can be expressed as C=f(w). Using units, explain the meaning of the following statements in terms of cars.

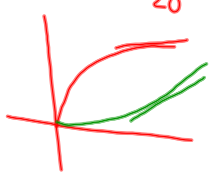
f(20)=100,000

The cost to produce 20 cars is \$100,000

f'(20)=3500 alternative notation

$$\left.\frac{dc}{dw}\right|_{w=20} = 3500$$

At the time the 20^{th} car is produced, the cost is increasing at a rate of $\frac{\$3500}{\text{car}}$.



INTERPRETATIONS OF THE DERIVATIVE

The cost of extracting T tons of ore from a copper mine is $C = f(T)$ dollars. What does it mean to say that $f'(2000) = 100$?

In the alternative notation,

$$f'(2000) = \left.\frac{dC}{dT}\right|_{T=2000} = 100.$$

Since C is measured in dollars and T is measured in tons, dC/dT must be measured in dollars per ton. So the statement $f'(2000) = 100$ says that when 2000 tons of ore have already been extracted from the mine, the cost of extracting the next ton is approximately \$100 more.

If $q = f(p)$ gives the number of pounds of sugar produced when the price per pound is p dollars, then what are the units and the meaning of the statement $f'(3) = 50$?

Since $f'(3)$ is the limit as $h \to 0$ of the difference quotient

$$\frac{f(3+h) - f(3)}{h},$$

the units of $f'(3)$ and the difference quotient are the same. Since $f(3+h) - f(3)$ is in pounds and h is in dollars, the units of the difference quotient and $f'(3)$ are pounds/dollar. The statement

$$f'(3) = 50 \text{ pounds/dollar}$$

tells us that the instantaneous rate of change of q with respect to p is 50 when $p = 3$. In other words, when the price is \$3, the quantity produced is increasing at 50 pounds/dollar. Thus, if the price increased by a dollar, the quantity produced would increase by approximately 50 pounds.

You are told that water is flowing through a pipe at a constant rate of 10 cubic feet per second. Interpret this rate as the derivative of some function.

You might think at first that the statement has something to do with the velocity of the water, but in fact a flow rate of 10 cubic feet per second could be achieved either with very slowly moving water through a large pipe, or with very rapidly moving water through a narrow pipe. If we look at the units—cubic feet per second—we realize that we are being given the rate of change of a quantity measured in cubic feet. But a cubic foot is a measure of volume, so we are being told the rate of change of a volume. One way to visualize this is to imagine all the water that is flowing through the pipe ending up in a tank somewhere. Let $V(t)$ be the volume of water in the tank at time t. Then we are being told that the rate of change of $V(t)$ is 10, or

$$V'(t) = \frac{dV}{dt} = 10. \quad \frac{ft^3}{sec}$$

Suppose $P = f(t)$ is the population of Mexico in millions, where t is the number of years since 1980. Explain the meaning of the statements:

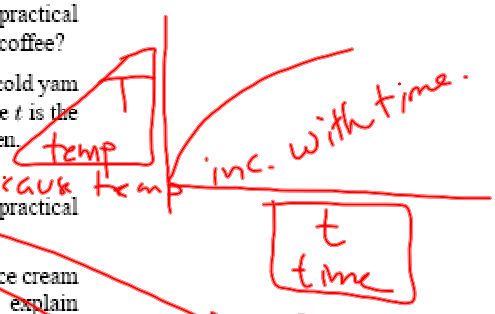
(a) $f'(6) = 2$ (b) $f^{-1}(95.5) = 16$ (c) $(f^{-1})'(95.5) = 0.46$

(a) The units of P are millions of people, the units of t are years, so the units of $f'(t)$ are millions of people per year. Therefore the statement $f'(6) = 2$ tells us that at $t = 6$ (that is, in 1986), the population of Mexico was increasing at 2 million people per year.
(b) The statement $f^{-1}(95.5) = 16$ tells us that the year when the population was 95.5 million was $t = 16$ (that is, in 1996).
(c) The units of the derivative $(f^{-1})'(P)$ are years per million of population. The statement $(f^{-1})'(95.5) = 0.46$ tells us that when the population was 95.5 million, it took about 0.46 years for the population to increase by 1 million.

Exercises

1. The temperature, H, in degrees Celsius, of a cup of coffee placed on the kitchen counter is given by $H = f(t)$, where t is in minutes since the coffee was put on the counter.
   (a) Is $f'(t)$ positive or negative? Give a reason for your answer.
   (b) What are the units of $f'(20)$? What is its practical meaning in terms of the temperature of the coffee?

2. The temperature, T, in degrees Fahrenheit, of a cold yam placed in a hot oven is given by $T = f(t)$, where t is the time in minutes since the yam was put in the oven.
   (a) What is the sign of $f'(t)$? Why? + because temp inc. with time.
   (b) What are the units of $f'(20)$? What is the practical meaning of the statement $f'(20) = 2$?



$\frac{°F}{min}$

At 20 minutes, the temp of yam is inc. at a rate of 2°F/min.

3. The cost, C (in dollars) to produce g gallons of ice cream can be expressed as $C = f(g)$. Using units, explain the meaning of the following statements in terms of ice cream.
   (a) $f(200) = 350$ (b) $f'(200) = 1.4$

4. The time for a chemical reaction, T (in minutes), is a function of the amount of catalyst present, a (in milliliters), so $T = f(a)$.
   (a) If $f(5) = 18$, what are the units of 5? What are the units of 18? What does this statement tell us about the reaction?
   (b) If $f'(5) = -3$, what are the units of 5? What are the units of -3? What does this statement tell us?

5. After investing \$1000 at an annual interest rate of 7% compounded continuously for t years, your balance is \$$B$, where $B = f(t)$. What are the units of dB/dt? What is the financial interpretation of dB/dt?

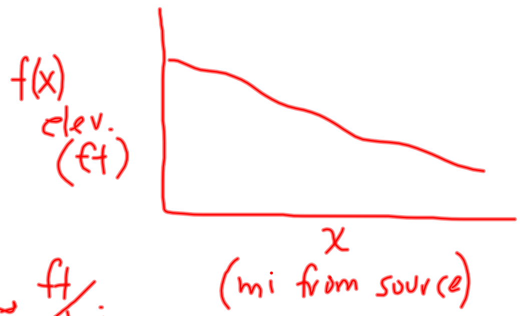
6. Investing \$1000 at an annual interest rate of r%, compounded continuously, for 10 years gives you a balance of \$$B$, where $B = g(r)$. Give a financial interpretation of the statements:
   (a) $g(5) \approx 1649$.
   (b) $g'(5) \approx 165$. What are the units of $g'(5)$?

7. Suppose $C(r)$ is the total cost of paying off a car loan borrowed at an annual interest rate of r%. What are the units of $C'(r)$? What is the practical meaning of $C'(r)$? What is its sign?

8. Suppose $P(t)$ is the monthly payment, in dollars, on a mortgage which will take t years to pay off. What are the units of $P'(t)$? What is the practical meaning of $P'(t)$? What is its sign?

9. Let $f(x)$ be the elevation in feet of the Mississippi river x miles from its source. What are the units of $f'(x)$? What can you say about the sign of $f'(x)$? neg

ft/mi



10. An economist is interested in how the price of a certain item affects its sales. At a price of \$$p$, a quantity, q, of the item is sold. If $q = f(p)$, explain the meaning of each of the following statements:
   (a) $f(150) = 2000$ (b) $f'(150) = -25$

17. If $g(v)$ is the fuel efficiency, in miles per gallon, of a car going at v miles per hour, what are the units of $g'(90)$? What is the practical meaning of the statement $g'(55) = -0.54$?

skip

18. (a) If you jump out of an airplane without a parachute, you fall faster and faster until wind resistance causes you to approach a steady velocity, called the *terminal* velocity. Sketch a graph of your velocity against time.
(b) Explain the concavity of your graph.
(c) Assuming wind resistance to be negligible at $t = 0$, what natural phenomenon is represented by the slope of the graph at $t = 0$?

Do 2.5 Exercises for Thursday (skip #17)

19. A company's revenue from car sales, C (in thousands of dollars), is a function of advertising expenditure, a, in thousands of dollars, so $C = f(a)$.
(a) What does the company hope is true about the sign of f'?
(b) What does the statement $f'(100) = 2$ mean in practical terms? How about $f'(100) = 0.5$?
(c) Suppose the company plans to spend about \$100,000 on advertising. If $f'(100) = 2$, should the company spend more or less than \$100,000 on advertising? What if $f'(100) = 0.5$?

20. Let $P(x)$ be the number of people of height $\leq x$ inches in the US. What is the meaning of $P'(66)$? What are its units? Estimate $P'(66)$ (using common sense). Is $P'(x)$ ever negative? [Hint: You may want to approximate $P'(66)$ by a difference quotient, using $h = 1$. Also,