Integration Techniques

Review first: Three ways to determine or approximate

$$\int_a^b f(x)\,dx \qquad \int_{-3}^{3} \sqrt{9-x^2}\,dx$$

1) Use the Fundamental Theorem (if possible...if you know the antiderivative)

2) Use your TI-89

3) Use approximations (Rectangles, sum seq, ...

4) Use geometry if possible.

We know how to find antiderivatives of elementary functions like sinx, cosx, e^x, x^n 1/x, polynomials, et

But what about $\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx$?

We'll see how to reverse the chain rule and product rules!

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx \overset{?}{=} -2\cos\sqrt{x} + C$$

$$2D_x(-\cos\sqrt{x})$$

$$=\cancel{2}(\sin\sqrt{x})\left(\frac{1}{\cancel{2}}x^{-\frac{1}{2}}\right)$$

$$x^{\frac{1}{2}}$$

First note:

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx$$

is a composition and the derivative of $\sqrt{x}$

is almost $\dfrac{1}{\sqrt{x}}$

So how could you antidifferentiate this?

Try $\int 3x^2\sqrt{5+x^3}\,dx$

In conjunction with the chain rule, there is an antidifferentiation property!

Let $\overbrace{F(g(x))}^{\text{composition}}+C$ be an antiderivative of some function. What function?

$D_x[F(g(x))+C] = F'(g(x))\cdot g'(x) = f(g(x))\cdot g'(x)$

Take the derivative via the chain rule

$$F'(g(x))\cdot g'(x)$$

$$= f(g(x))\cdot g'(x)$$

Sooooo...we can conclude that

$$\boxed{\int f(g(x))g'(x)dx=F(g(x))+C}$$

Comes straight from Chain Rule.

$$\int f(w)\,dw = F(w) + C$$

We have $\int f(g(x))\overbrace{g'(x)dx}=F(g(x))+C$

Let $w = g(x)$

$\frac{dw}{dx} = g'(x)$

$dw = g'(x)dx$

The application of this idea (the chain rule backwards) is easier if we make a change of variables, letting w=g(x). Then dw=g'(x)dx and

$$\int f(w)dw = F(w)+C$$

$$\int f(w)dw = F(w) + C$$

Where $w=g(x)$ and $dw=g'(x)dx$

This variable substitution process is known as the integration technique of SUBSTITUTION.

The technique: Reduce an indefinite integral to simpler form by trying to separate a composite function and the derivative of the inner function within the integrand. Set w=inner function, or the function whose derivative appears elsewhere.

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$$

Let $w = \boxed{\sqrt{x}} = x^{\frac{1}{2}}$

$dw = \frac{1}{2}\boxed{x^{-\frac{1}{2}}dx}$

$2dw = \frac{1}{\sqrt{x}}dx$

$$2\int \sin w\, dw = 2(-\cos w) + C$$

$$= -2\cos\sqrt{x} + C$$

Never mix variables!

$\cancel{\int \frac{dx}{w}}$

Never want a differential (dx) in denom.

$\cancel{\int \frac{w}{dw}}$

Examples

1) $\int \sqrt[3]{x-1}\,dx = \int \sqrt[3]{w}\,dw = \frac{3}{4}w^{4/3} + C$

$(w^{1/3})$

$w = x-1$

$dw = dx$

$= \frac{3}{4}(x-1)^{4/3} + C$

2) $\int \sqrt[3]{2x-1}\,dx = \frac{1}{2}\int \sqrt[3]{w}\,dw = \frac{1}{2}\cdot\frac{3}{4}w^{4/3} + C$

$w = 2x-1$

$dw = 2dx$

$\frac{1}{2}dw = dx$

$= \frac{3}{8}(2x-1)^{4/3} + C$

3) $\int x^4 \cdot \sqrt[3]{x^5-1}\,dx = \frac{1}{5}\int \sqrt[3]{w}\,dw$

$w^{1/3}$

$w = x^5-1$

$dw = 5x^4\,dx$

$\frac{1}{5}dw = x^4\,dx$

$= \frac{1}{5}\cdot\frac{3}{4}w^{4/3} + C$

$= \frac{3}{20}(x^5-1)^{4/3} + C$

Examples (cont)

belong with y

4) $\int_0^1 y(y^2+5)^4\,dy = \frac{1}{2}\int_5^6 w^4\,dw = \frac{1}{2}\cdot\frac{1}{5}w^5\Big|_5^6$

$w = y^2+5$

$dw = 2y\,dy$

$\frac{1}{2}dw = y\,dy$

$= \frac{1}{10}\left[6^5 - 5^5\right]$

5) $\int te^{t^2+1}\,dt = \frac{1}{2}\int e^w\,dw = \frac{1}{2}e^w + C$

$w = t^2+1$

$dw = 2t\,dt$

$\frac{1}{2}dw = t\,dt$

$= \frac{1}{2}e^{t^2+1} + C$

6) $\int \frac{e^m}{4+e^m}\,dm = \int \frac{1}{w}\,dw = \ln|w| + C$

$w = 4+e^m$

$dw = e^m\,dm$

$= \ln|4+e^m| + C$

Examples (cont)

* Know this one

7) $\int \tan(p)dp = \int \frac{\sin p}{\cos p}dp = -\int \frac{1}{w}dw = -\ln|w| + C$

$w = \cos p$

$dw = -\sin p\,dp$

$-dw = \sin p\,dp$

$= -\ln|\cos p| + C$

$= \ln\left|\frac{1}{\cos p}\right| + C$

$= \ln|\sec p| + C$

8) $\int t^2\sqrt{1+t}\,dt = \int (w-1)^2\sqrt{w}\,dw =$

$w = 1+t \Rightarrow t = w-1$

$dw = dt \qquad t^2 = (w-1)^2$

$\int (w^2-2w+1)\sqrt{w}\,dw$

$\int w^{5/2} - 2w^{3/2} + w^{1/2}\,dw$

...

9) $\int_0^1 x\sqrt[3]{(4-x^2)}dx = -\frac{1}{2}\int_4^3 \sqrt{w}\,dw =$ ...

$w = 4-x^2$

$dw = -2x\,dx$

$-\frac{1}{2}dw = x\,dx$

10) $\int(\sin^4 t)(\cos t)dt = \int w^4\,dw = \frac{1}{5}w^5 + C$

$= \frac{1}{5}(\sin^5 t) + C$

$w = \sin t$

$dw = \cos t\,dt$

11) $\int(\sin^4 t^3)(t^2\cos t^3)dt = \frac{1}{3}\int w^4\,dw$ ...

$w = \sin t^3$

$dw = (\cos t^3)\cdot 3t^2\,dt$

12) $\int x\sqrt[3]{x+4}\,dx$

13) $\int \frac{dt}{t^{\frac{1}{3}} - t}$

14) $\int \sqrt{1+\sqrt{x}}\,dx$

$$\int \sqrt{1+\sqrt{x}}\,dx$$

Find the area enclosed by the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$