

# Integration Techniques



Review first: Three ways to determine or approximate

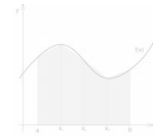
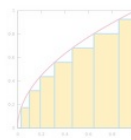
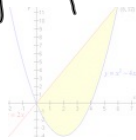
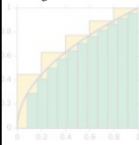
$$\int_a^b f(x) dx \quad \int_{-3}^3 \sqrt{9-x^2} dx$$

1) Use the Fundamental Theorem (if possible... if you know the antiderivative)

2) Use your TI-89

3) Use approximations (Rectangles, sum seq, ...)

4) Use geometry if possible.

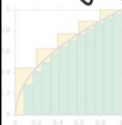


We know how to find antiderivatives of elementary functions like  $\sin x$ ,  $\cos x$ ,  $e^x$ ,  $x^n$ ,  $1/x$ , polynomials, etc

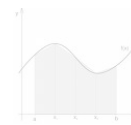
But what about  $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$  ?

We'll see how to reverse the chain rule and product rules!

$$\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx = -2 \cos \sqrt{x} + C$$



$$2 D_x(\cos \sqrt{x}) = 2(\sin \sqrt{x}) \left( \frac{1}{2} x^{-\frac{1}{2}} \right)$$





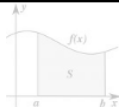
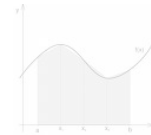
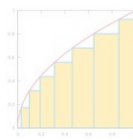
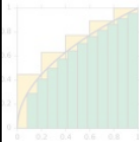
First note:

$$\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$$

is a composition and the derivative of  $\sqrt{x}$

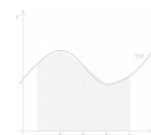
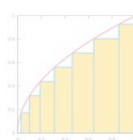
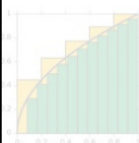
is almost  $\frac{1}{\sqrt{x}}$

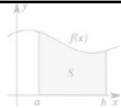
So how could you antidifferentiate this?



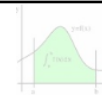
Try  $\int 3x^2 \sqrt{5+x^3} dx$

In conjunction with the chain rule, there is an antidifferentiation property!





composition



Let  $F(g(x)) + C$  be an antiderivative of some function  
What function?

$$D_x[F(g(x)) + C] = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x)$$

Take the derivative via the chain rule

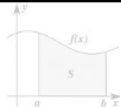
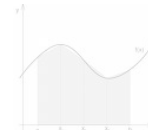
$$F'(g(x)) \cdot g'(x)$$

$$= f(g(x)) \cdot g'(x)$$

Sooooo...we can conclude that

$$\int f(g(x))g'(x)dx = F(g(x)) + C$$

comes straight from Chain Rule.



$$\int f(w)dw = F(w) + C$$

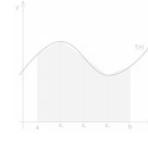
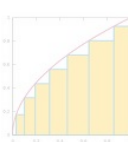
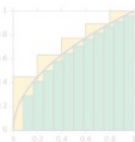


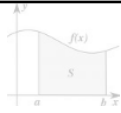
We have  $\int f(g(x))g'(x)dx = F(g(x)) + C$

$$\begin{aligned} \text{Let } w &= g(x) \\ \frac{dw}{dx} &= g'(x) \\ dw &= g'(x)dx \end{aligned}$$

The application of this idea (the chain rule backwards) is easier if we make a change of variables, letting  $w=g(x)$ . Then  $dw=g'(x)dx$  and

$$\int f(w)dw = F(w) + C$$





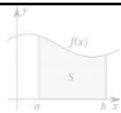
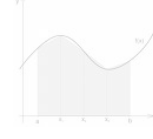
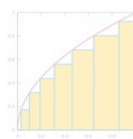
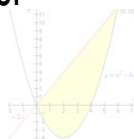
$$\int f(w)dw = F(w) + C$$



Where  $w=g(x)$  and  $dw=g'(x)dx$

This variable substitution process is known as the integration technique of SUBSTITUTION.

The technique: Reduce an indefinite integral to simpler form by trying to separate a composite function and the derivative of the inner function within the integrand. Set  $w$ =inner function, or the function whose derivative appears elsewhere.



$$\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$$

Let  $w = \sqrt{x} = x^{\frac{1}{2}}$   
 $dw = \frac{1}{2} x^{-\frac{1}{2}} dx$

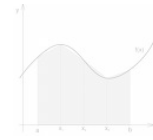
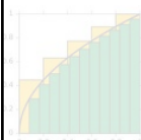
$$2dw = \frac{1}{\sqrt{x}} dx$$

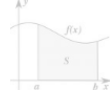
$$2 \int \sin w dw = 2(-\cos w) + C$$

$$= 2(\cos \sqrt{x} + C)$$


Never mix variables!

~~$\int \frac{dx}{w}$~~   
 Never want a derivative (dx) in denom.





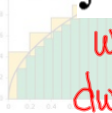


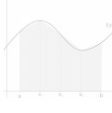

## Examples




1)  $\int \sqrt[3]{x-1} dx = \int \sqrt[3]{w} dw = \frac{3}{4} w^{\frac{4}{3}} + C$   
 $w = x-1$   
 $dw = dx$   
 $(w^{\frac{4}{3}}) = \frac{3}{4} (x-1)^{\frac{4}{3}} + C$

2)  $\int \sqrt[3]{2x-1} dx = \frac{1}{2} \int \sqrt[3]{w} dw = \frac{1}{2} \cdot \frac{3}{4} w^{\frac{4}{3}} + C$   
 $w = 2x-1$   
 $dw = 2dx$   
 $\frac{1}{2} dw = dx$   
 $= \frac{3}{8} (2x-1)^{\frac{4}{3}} + C$

3)  $\int x^4 \cdot \sqrt[3]{x^5-1} dx = \frac{1}{5} \int \sqrt[3]{w} dw$   
 $w = x^5-1$   
 $dw = 5x^4 dx$   
 $\frac{1}{5} dw = x^4 dx$   
 $= \frac{1}{5} \cdot \frac{3}{4} w^{\frac{4}{3}} + C$   
 $= \frac{3}{20} (x^5-1)^{\frac{4}{3}} + C$

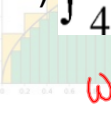
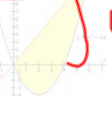

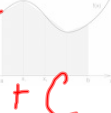
## Examples (cont)



4)  $\int_0^1 y(y^2+5)^4 dy = \frac{1}{2} \int_5^6 w^4 dw = \frac{1}{2} \cdot \frac{1}{5} w^5 \Big|_5^6$   
 $w = y^2+5$   
 $dw = 2y dy$   
 $\frac{1}{2} dw = y dy$   
 $= \frac{1}{10} [6^5 - 5^5]$

5)  $\int te^{t^2+1} dt = \frac{1}{2} \int e^w dw = \frac{1}{2} e^w + C$   
 $w = t^2+1$   
 $dw = 2t dt$   
 $\frac{1}{2} dw = t dt$   
 $= \frac{1}{2} e^{t^2+1} + C$

6)  $\int \frac{e^m}{4+e^m} dm = \int \frac{1}{w} dw = \ln|w| + C$   
 $w = 4+e^m$   
 $dw = e^m dm$   
 $= \ln|4+e^m| + C$

# Examples (cont)

\* Know this one.

$$7) \int \tan(p) dp = \int \frac{\sin p}{\cos p} dp = - \int \frac{1}{w} dw = -\ln|w| + C$$

$$w = \cos p$$

$$dw = -\sin p dp$$

$$-dw = \sin p dp$$

$$= -\ln|\cos p| + C$$

$$= \ln\left|\frac{1}{\cos p}\right| + C$$

$$= \ln|\sec p| + C$$

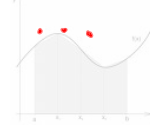
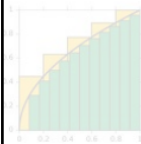
$$8) \int t^2 \sqrt{1+t} dt = \int (w-1)^2 \sqrt{w} dw =$$

$$w = 1+t \Rightarrow t = w-1$$

$$dw = dt \quad t^2 = (w-1)^2$$

$$\int (w^2 - 2w + 1) \sqrt{w} dw$$

$$\int w^{5/2} - 2w^{3/2} + w^{1/2} dw$$



$$9) \int_0^1 x^3 \sqrt{4-x^2} dx = \frac{-1}{2} \int_4^3 \sqrt{w} dw =$$

$$w = 4-x^2$$

$$dw = -2x dx$$

$$-\frac{1}{2} dw = x dx$$

$$10) \int (\sin^4 t)(\cos t) dt = \int w^4 dw = \frac{1}{5} w^5 + C$$

$$w = \sin t$$

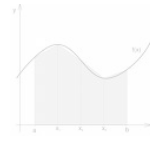
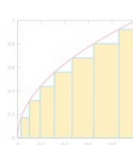
$$dw = \cos t dt$$

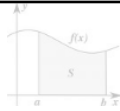
$$= \frac{1}{5} (\sin^5 t) + C$$

$$11) \int (\sin^4 t^3)(t^2 \cos t^3) dt = \frac{1}{3} \int w^4 dw \dots$$

$$w = \sin t^3$$

$$dw = (\cos t^3) \cdot 3t^2 dt$$

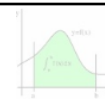
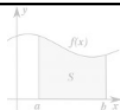
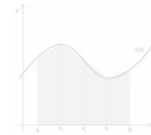
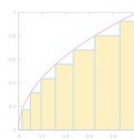
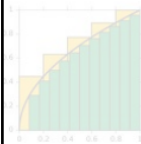




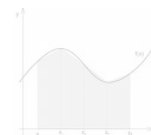
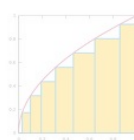
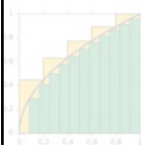
$$12) \int x^3 \sqrt{x+4} dx$$

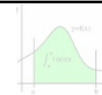
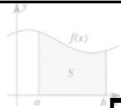
$$13) \int \frac{dt}{t^3 - t}$$

$$14) \int \sqrt{1 + \sqrt{x}} dx$$



$$\int \sqrt{1 + \sqrt{x}} dx$$





Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

