Daily Quiz

Come in at 11:40 to make up

6)

Integration Technique #4

TRIGONOMETRIC SUBSTITUTION

A technique to handle integrands having form:

$(a^2+x^2)^r$ $(x^2-a^2)^r$ $(a^2-x^2)^r$

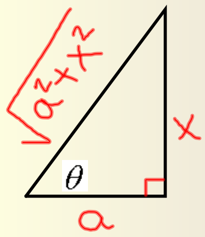
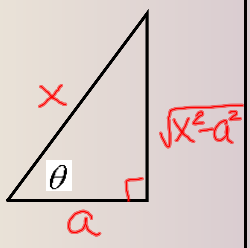
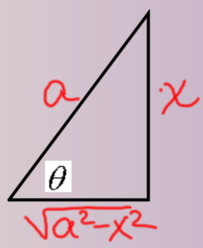
where a is a positive real number and r is rational

$\int$ complex algebraic form like above $\Rightarrow$ $\int$ trigonometric

MODELS

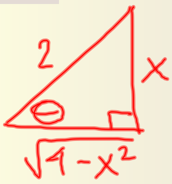
Make triangles to convert algebraic to trigonometric forms.

we prefer not to have $\boxed{x=\text{co-function}}$

$(a^2+x^2)^r$	$(x^2-a^2)^r$	$(a^2-x^2)^r$
		
$x=a\tan\theta$	$x=a\sec\theta$	$x=a\sin\theta$
$dx=a\sec^2\theta\,d\theta$	$dx=a\sec\theta\tan\theta\,d\theta$	$dx=a\cos\theta\,d\theta$

Examples

1) $$\int\frac{dx}{(4-x^2)^{\frac{3}{2}}}=\int\frac{2\cos\theta\,d\theta}{8\cos^3\theta}=\frac{1}{4}\int\sec^2\theta\,d\theta$$



$$=\frac{1}{4}\tan\theta+C$$

$$=\frac{1}{4}\frac{x}{\sqrt{4-x^2}}+C$$

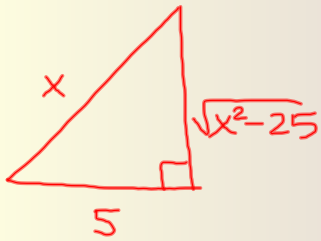
$x=2\sin\theta$

$dx=2\cos\theta\,d\theta$

$\sqrt{4-x^2}=2\cos\theta$

$\therefore (4-x^2)^{\frac{3}{2}}=8\cos^3\theta$

2) $\displaystyle\int \frac{dx}{\sqrt{x^2-25}} = \int \frac{\cancel{5}\sec\theta\,\cancel{\tan\theta}\,d\theta}{\cancel{5}\,\cancel{\tan\theta}}$



$= \ln|\sec\theta + \tan\theta| + C$

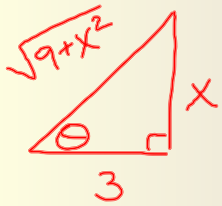
$= \ln\left|\frac{x}{5} + \frac{\sqrt{x^2-25}}{5}\right| + C$

$x = 5\sec\theta$

$dx = 5\sec\theta\tan\theta\,d\theta$

$\sqrt{x^2-25} = 5\tan\theta$

3) $\displaystyle\int \frac{dx}{\sqrt{(9+x^2)}} = \int \frac{3\sec^2\theta\,d\theta}{3\sec\theta}$



$= \int \sec\theta\,d\theta$

$x = 3\tan\theta$

$dx = 3\sec^2\theta\,d\theta$

$\sqrt{9+x^2} = 3\sec\theta$

$= \ln|\sec\theta + \tan\theta| + C$

$= \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C$

3.5) $\int x\sqrt{x^2-36}\,dx$

$w = x^2 - 36$

$dw = 2x\,dx$

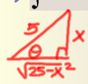
$\frac{1}{2}dw = x\,dx$

$$\frac{1}{2}\int \sqrt{w} \quad dx$$

$$= \frac{1}{2}\cdot\frac{2}{3}w^{3/2} + C$$

$$= \frac{1}{3}(x^2-36)^{3/2} + C$$

4) $\int x^2\sqrt{25-x^2}\,dx = \int 25\sin^2\theta \cdot 5\cos\theta \cdot 5\cos\theta\, d\theta$



$x = 5\sin\theta$

$dx = 5\cos\theta\, d\theta$

$x^2 = 25\sin^2\theta$

$\sqrt{25-x^2} = 5\cos\theta$

$$= 625\int \sin^2\theta\cos^2\theta\, d\theta$$

$$= \frac{625}{4}\int (1-\cos 2\theta)(1+\cos 2\theta)\, d\theta$$

$$= \frac{625}{4}\int (1-\cos^2 2\theta)\, d\theta$$

$$= \frac{625}{4}\int \sin^2 2\theta \; d\theta$$

$$= \frac{625}{4}\int \frac{1-\cos 4\theta}{2}\, d\theta$$

$$\frac{625}{8}\int (1-\cos 4\theta)\, d\theta$$

$$\frac{625}{8}\left[\theta - \frac{\sin 4\theta}{4}\right] + C$$

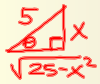
$$\frac{625}{8}\left[\sin^{-1}\frac{x}{5} - \frac{2\sin 2\theta\cos 2\theta}{4}\right] + C$$

$$\frac{625}{8}\left[\sin^{-1}\frac{x}{5} \; - \; \frac{\cancel{2}\cdot 2\sin\theta\cos\theta(1-2\sin^2\theta)}{\cancel{4}}\right]$$

$$\left[\frac{x}{5}\,\frac{\sqrt{25-x^2}}{5}\left(1-\frac{2x^2}{25}\right)\right]$$

$+C$

4) $\int x^2\sqrt{25-x^2}\,dx = \int 25\sin^2\theta \cdot 5\cos\theta \cdot 5\cos\theta\, d\theta$



$x = 5\sin\theta$

$dx = 5\cos\theta\, d\theta$

$\sqrt{25-x^2} = 5\cos\theta$

$= 625\int \sin^2\theta\cos^2\theta\, d\theta$

$\cos 2\theta = 1 - 2\sin^2\theta$

$= 625\int \left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{1+\cos 2\theta}{2}\right) d\theta$

$= \frac{625}{4}\int \left(1-\cos^2 2\theta\right) d\theta$

$= \frac{625}{4}\int \left(1-\left(\frac{1+\cos 4\theta}{2}\right)\right) d\theta$

$= \frac{625}{4}\int \left(\frac{2-1-\cos 4\theta}{2}\right) d\theta$

$= \frac{625}{8}\int \left(1-\cos 4\theta\right) d\theta$

$= \frac{625}{8}\left(\theta - \frac{\sin 4\theta}{4}\right) + C$

$= \frac{625}{8}\left(\sin^{-1}\frac{x}{5} - \left(\frac{x\sqrt{25-x^2}}{25}\left(1-\frac{2x^2}{25}\right)\right)\right) + C$

yeah, it's ugly

$\frac{x}{5} = \sin\theta$

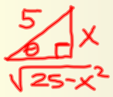
$\sin^{-1}\frac{x}{5} = \theta$

$\sin(4\theta) = \sin 2(2\theta) = 2\sin 2\theta\cos 2\theta = 2(2\sin\theta\cos\theta)(1-2\sin^2\theta)$

$4\left(\frac{x}{5}\cdot\frac{\sqrt{25-x^2}}{5}\right)\left(1-2\cdot\left(\frac{x^2}{25}\right)\right)$

Alternate Solution

4) $\int x^2\sqrt{25-x^2}\,dx = \int 25\sin^2\theta \cdot 5\cos\theta \cdot 5\cos\theta\, d\theta$



$x = 5\sin\theta$

$dx = 5\cos\theta\, d\theta$

$\sqrt{25-x^2} = 5\cos\theta$

$= 625\int \sin^2\theta\cos^2\theta\, d\theta$

$= 625\int \left(\sin\theta\cos\theta\right)^2 d\theta$

$= 625\int \left(\frac{\sin 2\theta}{2}\right)^2 d\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

$= \frac{625}{4}\int \sin^2 2\theta\, d\theta$

$= \frac{625}{4}\int \frac{1-\cos 4\theta}{2}\, d\theta$

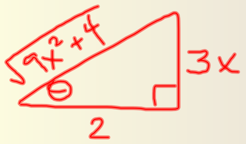
$= \frac{625}{8}\int \left(1-\cos 4\theta\right) d\theta$

$= \frac{625}{8}\left[\theta - \frac{1}{4}\sin 4\theta\right] + C$

see last page ü

5) $\int \frac{1}{(9x^2+4)^{\frac{3}{2}}}dx$

5) $\int \frac{1}{(9x^2+4)^{\frac{3}{2}}}dx = \int \frac{1}{8\sec^3\theta} \cdot \frac{2}{3}\sec^2\theta \, d\theta$



$\frac{3x}{2} = \tan\theta$

$x = \frac{2}{3}\tan\theta$

$dx = \frac{2}{3}\sec^2\theta \, d\theta$

$\sqrt{9x^2+4} = 2\sec\theta$

$(9x^2+4)^{3/2} = 8\sec^3\theta$

$= \frac{1}{12}\int \frac{1}{\sec\theta} d\theta$

$= \frac{1}{12}\int \cos\theta \, d\theta$

$= \frac{1}{12}\sin\theta + C$

$= \frac{1}{12} \cdot \frac{3x}{\sqrt{9x^2+4}} + C$

$= \frac{x}{4\sqrt{9x^2+4}} + C$